Massive random matrix ensembles at $\beta = 1$ & 4 : QCD in three dimensions

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The zero momentum sectors in effective theories of three dimensional QCD coupled to pseudo-real (two colors) and real (adjoint) quarks in a classically parity-invariant manner have alternative descriptions in terms of orthogonal and symplectic ensembles of random matrices. Using this correspondence, we compute finite-volume QCD partition functions and correlation functions of Dirac operator eigenvalues in a presence of finite quark masses of the order of the smallest Dirac eigenvalue. These novel correlation functions, expressed in terms of quaternion determinants, are reduced to conventional results for the Gaussian ensembles in the quenched limit.

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I. INTRODUCTION

Spontaneous breaking of global symmetries has long been a subject of an extensive study for quantum field theories in various dimensions. Besides well-known cases such as 2D Gross-Neveu and Schwinger models and 4D QCD that exhibit breakdown of discrete or continuous chiral symmetry, it has been suspected that 3D QED and QCD may also undergo spontaneous breaking of flavor symmetry [1–4]. For such odd dimensional field theories, there is no comprehensive theorem that predicts the surviving part of the flavor symmetry as in the case of even dimensions [5], as the notion of fermion chirality is absent. However, if there are even $N_f = 2n$ number of massless 2-component complex fermions ψ_j, one can group them appropriately into 4-component complex fermions Ψ_j [2–4]:

$$\mathcal{L} = \sum_{j=1}^{2n} \bar{\psi}_j D_\mu \sigma_\mu \psi_j = \sum_{j=1}^{n} \bar{\Psi}_j D_\mu \gamma_\mu \Psi_j, \qquad (1.1)$$

where $\gamma_\mu = \sigma_\mu \otimes \sigma_3$ ($\mu = 1, 2, 3$) in Euclidean 3D space. Then one can define a *quasi*-chirality according to two Hermitian matrices, e.g. $\gamma_4 = \mathbb{1} \otimes \sigma_1$ and $\gamma_5 = \mathbb{1} \otimes \sigma_2$, that anticommute with the Dirac operator $D_\mu \gamma_\mu$ and with each other. Generators of the flavor $U(2n)$ group are rearranged into that of $U(n)$ group times $\{1, \gamma_4, \gamma_5, \gamma_S = \mathbb{1} \otimes \sigma_3\}$ in the 4-component notation. In order to predict spontaneous flavor symmetry breaking pattern along the same line as in 4D QCD, one introduces a small symmetry-breaking mass term that is parity-invariant, i.e., $\sum_{j=1}^{n} m_j \bar{\Psi}_j \Psi_j$ but not $\sum_{j=1}^{n} m_j \bar{\Psi}_j \gamma_S \Psi_j$. In the 2-component notation it leads to include masses $\{m\} = (m_1, \ldots, m_n, -m_1, \ldots, -m_n)$. As the fermion determinant

$$\prod_{j=1}^{n} \det(iD_\mu \gamma_\mu + im_j) = \prod_{j=1}^{n} \det(-(D_\mu \sigma_\mu)^2 + m_j^2) \quad (1.2)$$

is positive definite under this restriction, one can appeal to the Vafa-Witten theorem [5] and predict that if the flavor symmetry is spontaneously broken, the absolute values of the fermion condensate $\langle \bar{\psi}_j \psi_j \rangle$ are equal for all $j = 1, \ldots, 2n$ and their signs are the same as those of respective masses. That is, the continuous part of the global symmetry group is broken according to [4,6]

$$U(2n) \to U(n) \times U(n) \qquad (1.3)$$

by the order parameter

$$\Sigma = \frac{1}{2n} \sum_{j=1}^{2n} |\langle \bar{\psi}_j \psi_j \rangle|, \qquad (1.4)$$

while the discrete $\mathbf{Z}_2$ group (a product of the parity and the exchange of fields $\psi_j \leftrightarrow \psi_{n+j}$) remains unbroken. The formation of the quark condensate was indeed observed in Monte-Carlo simulations on a lattice [7]. This peculiar pattern of flavor symmetry breaking can also be predicted for 3D large-N_c QCD by the Coleman-Witten argument [8], as remarked in Ref. [6]. One can repeat the above argument valid for complex (fundamental representation of $SU(N_c \geq 3)$ gauge group) fermions towards the cases with even flavors of pseudoreal (fundamental representation of $SU(2)$ gauge group) and of real (adjoint representation of $SU(N_c)$ gauge group) fermions. We assign Dyson indices $\beta = 2, 1, 4$, respectively, to these three cases, according to the anti-unitary symmetries of the associated Dirac operators [9]. Then the continuous parts of the global symmetry groups are predicted to be broken down as [10]

$$Sp(2n) \to Sp(n) \times Sp(n) \qquad (\beta = 1), \qquad (1.5a)$$
$$SO(2n) \to SO(n) \times SO(n) \quad (\beta = 4). \qquad (1.5b)$$

These symmetry breaking patterns determine the forms of the low-energy effective Lagrangian of associated Nambu-Goldstone bosons:

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4} \operatorname{tr} \partial_\mu U \partial_\mu U^\dagger - \Sigma \operatorname{tr} \boldsymbol{m} U \Gamma U^\dagger + \cdots, \qquad (1.6)$$

where $U(\mathbf{x})$ takes its value in the coset manifolds[†]

$$\mathcal{M}_F = \text{AIII}_{n,n}, \ \text{CII}_{n,n}, \ \text{BDI}_{n,n}, \qquad (1.7)$$

respectively for $\beta = 2, 1, 4$. The mass and chiral matrices in the above are defined as

$$\boldsymbol{m} = \begin{cases} \operatorname{diag}(m_1, \ldots, m_n) \otimes \sigma_3 & (\beta = 2, 4) \\ \operatorname{diag}(m_1, \ldots, m_n) \otimes \sigma_3 \otimes i\sigma_2 & (\beta = 1) \end{cases}, \quad (1.8)$$

$$\Gamma = \begin{cases} \mathbb{1}_n \otimes \sigma_3 & (\beta = 2, 4) \\ \mathbb{1}_n \otimes \sigma_3 \otimes i\sigma_2 & (\beta = 1) \end{cases}. \qquad (1.9)$$

Given non-linear σ models (NLσMs) in 4D, Leutwyler and Smilga [11] proposed to extract out of them nonperturbative, exact information on the spectra of Dirac operators by imposing a constraint on the parameters: the linear dimension L of the system be much smaller than the Compton length of Nambu-Goldstone bosons $\sim F/\sqrt{m\Sigma}$. In this 'ergodic' regime where the zero momentum mode of $U(\mathbf{x})$ dominates, the effective finite-volume partition function simplifies into a finite-dimensional integral and its small-m expansion yields a sequence of spectral sum rules. Verbaarschot and collaborators [12,13] (see Ref. [14] for a exhaustive list of references) made an important observation along this line, that the 0D finite-volume partition functions could as well be derived from models much simpler than QCD, random matrix ensembles (RMEs). In our 3D QCD context [6,10], it means that in the limit

$$L \to \infty, \quad m_i \to 0, \quad \mu_i \equiv L^3 \Sigma m_i : \text{fixed}, \qquad (1.10)$$

the finite-volume partition functions

$$\mathcal{Z}(\{\mu\}) = \int_{\mathcal{M}_F} dU \, \exp\left(\operatorname{tr} \boldsymbol{\mu} U \Gamma U^\dagger \right), \qquad (1.11)$$

$$\boldsymbol{\mu} \equiv L^3 \Sigma \boldsymbol{m},$$

have alternative representation in terms of large-N RMEs

$$Z(\{m\}) = \int_{\mathcal{D}} dH \mathrm{e}^{-\beta \operatorname{tr} V(H^2)} \prod_{i=1}^n \det\left(H^2 + m_j^2 \right), \quad (1.12)$$

where the integral domains $\mathcal{D}$ are sets of $N \times N$ complex hermitian ($u(N) = T(\mathrm{A}_N)$), real symmetric ($o(N) = T(\mathrm{AI}_N)$), and quaternion selfdual ($sp(N) = T(\mathrm{AII}_N)$) matrices H for $\beta = 2, 1, 4$, respectively. The determinant in the $\beta = 4$ case is understood as a quaternion determinant (Tdet). Their proofs consist of the 'color-flavor' (or Hubbard-Stratonovich) transformation [15,16] that converts the integration variables into matrices with small ($n \times n$) dimensions, and the saddle point method under which

$$N \to \infty, \quad m_i \to 0, \quad \mu_i \equiv \pi \bar\rho(0) m_i : \text{fixed.} \qquad (1.13)$$

Here $\bar\rho(0)$ stands for the large-N spectral density of the random matrix H:

$$\bar\rho(x) = \lim_{N \to \infty} \langle \operatorname{tr} \delta(x - H) \rangle, \qquad (1.14)$$

at the spectral origin. The RMEs (1.12) are motivated by the microscopic theories (3D Euclidean QCD) on a lattice, with a crude simplification of replacing matrix elements of the Hermitian Dirac operator $i\slashed{D} = (i\partial_\mu + A_\mu)\sigma_\mu$ by random numbers H_{jk} distributed according to the weight $\mathrm{e}^{-\beta \operatorname{tr} V(H^2)}$. Under this correspondence, the microscopic limit (1.13) is equivalent to Leutwyler-Smilga limit (1.10), since the size N of the matrix H is interpreted as the number of sites L^3 of the lattice on which QCD is discretized, and the Dirac spectral density at zero virtuality $\bar\rho(0)$ is related to the quark condensate by the Banks-Casher relation $\Sigma = \pi\bar\rho(0)/L^3$ [17].

On the other hand, the situation is far more subtle in the case with an odd number ($N_f = 2n + 1$) of 2-component fermion flavors [3,6]. In the massless case, the fermion determinant $\det^{2n+1}(iD_\mu \sigma_\mu)$ is not positive definite, and its phase is gauge dependent. This dependence can be compensated by the Chern-Simons term that is yielded by a gauge-invariant regularization, but this anomalous term explicitly breaks the parity [3]. Therefore, even in a presence of small masses $\{m\} = (m_1, \ldots, m_n, -m_1, \ldots, -m_n, 0)$ that respects the $\boldsymbol{Z}_2$ invariance (combining parity and flavor exchange) classically, one cannot appeal to the previous argument to derive low-energy effective Lagrangians. With this situation in mind, we nevertheless adopt a pattern of spontaneous flavor symmetry breaking proposed by Verbaarschot and Zahed [6] for $\beta = 2$ and its generalizations to $\beta = 1$ and 4:

$$U(2n+1) \to U(n) \times U(n+1) \qquad (\beta = 2), \quad (1.15a)$$
$$Sp(2n+1) \to Sp(n) \times Sp(n+1) \qquad (\beta = 1), \quad (1.15b)$$
$$SO(2n+1) \to SO(n) \times SO(n+1) \qquad (\beta = 4), \quad (1.15c)$$

leading to NLσMs of Nambu-Goldstone fields over the coset manifolds

$$\mathcal{M}_F = \text{AIII}_{n,n+1}, \ \text{CII}_{n,n+1}, \ \text{BDI}_{n,n+1}, \qquad (1.16)$$

respectively. Then, by the same token as in the case of even N_f, one can write down corresponding RMEs [18]:

$$Z(\{m\}) = \int_{\mathcal{D}} dH \mathrm{e}^{-\beta \operatorname{tr} V(H^2)} \det H \prod_{i=1}^n \det\left(H^2 + m_j^2 \right), \qquad (1.17)$$

which are equivalent, in the limit (1.13), to the 'finite-volume partition functions' [6]

$$\mathcal{Z}(\{\mu\}) = \int_{\mathcal{M}_F} dU \begin{cases} \cosh\left(\operatorname{tr} \boldsymbol{\mu} U \Gamma U^\dagger \right) & (N : \text{even}) \\ \sinh\left(\operatorname{tr} \boldsymbol{\mu} U \Gamma U^\dagger \right) & (N : \text{odd}) \end{cases}, \quad (1.18)$$

$$\Gamma = \begin{cases} \operatorname{diag}(\mathbb{1}_n, -\mathbb{1}_{n+1}) & (\beta = 2, 4) \\ \operatorname{diag}(\mathbb{1}_n, -\mathbb{1}_{n+1}) \otimes i\sigma_2 & (\beta = 1) \end{cases}. \quad (1.19)$$

As the Chern-Simons term cannot be incorporated within these RMEs, their physical relevance is unclear [6]. An immediate problem is that if the rank N of the matrix H is odd, the partition function (1.17) (or (1.18)) is zero, and the (unnormalized) correlation functions are odd under a simultaneous change of signs of the arguments, which are unacceptable as physical observables. The above relationships between RMEs and NLσMs for 3D QCD consist, together with its counterpart for 4D QCD ($\mathcal{M}_F = A_n, AII_n, AI_n$ and $\mathcal{D} = T(AIII_{N,N'}), T(BDI_{N,N'}), T(CII_{N,N'})$ for $\beta = 2, 1, 4$, respectively), a part of Zirnbauer's complete classification scheme of RMEs in terms of Riemannian symmetric spaces [19].

These RMEs are technically suited for the computation of correlations of eigenvalues $\{x\}$ of the Dirac operator $i\not{D} \sim H$ in the microscopic asymptotic limit where the energy eigenvalues are scaled as the quark masses are in Eq.(1.13),

$$N \to \infty, \quad x \to 0, \quad \lambda = \pi\bar{\rho}(0)x : \text{fixed.} \qquad (1.20)$$

For the chiral RMEs describing 4D QCD in the low-energy ergodic regime, such Dirac spectral correlators have been computed previously for the massless [13,20–24] and recently for the massive cases [25–31] with all three values of β. On the other hand, for the (non-chiral) RMEs describing 3D QCD, they have been analytically computed solely for the unitary ($\beta = 2$) ensemble, in the massless case [20,6], as well as in the massive case [32,18]. For other values of β, a numerical work based on a finite-N formula for the correlation functions recently appeared only in the massless case [33]. The subject of this Article is to complete this program by analytically computing the partition and correlation functions for the

orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) ensembles in a presence of finite scaled mass parameters. We employ a slightly modified version of the method used in our previous articles [29,30].

We finally remark on the universality issue. It was noticed by Şener and Verbaarschot [34] (see also Ref. [35]) and proved by Widom [36] that the diagonal element $S(x, y)$ of the quaternion kernel for an orthogonal or symplectic ensemble, and accordingly all spectral correlation functions thereof, can be constructed from the scalar kernel $K(x, y)$ for a unitary ensemble sharing the same weight function:

$$S^T = \left(I - (I - K)\varepsilon K D\right)^{-1} K \qquad (\beta = 1), \qquad (1.21a)$$

$$S = \left(I - (I - K)D K \varepsilon\right)^{-1} K \qquad (\beta = 4), \qquad (1.21b)$$

where I, D, ε, S, K stand for integral operators with convolution kernels $\delta(x - y)$, $\delta'(x - y)$, $\frac{1}{2}\text{sgn}(x - y)$, $S(x, y)$, $K(x, y)$, respectively, ^{T} and $^{-1}$ stand for transpose and inverse operators. Since the scalar kernel in the asymptotic limit (1.20), (1.13) is insensitive to the details of the potential $V(x^2)$ either in the absence [37,38] or in the presence of finite and nonzero μ's [32,18], the universality of correlation functions for orthogonal and symplectic ensembles are automatically guaranteed[‡]. Therefore it suffices for us to concentrate onto Gaussian ensembles, $V(x^2) = x^2/2$. This choice leads to Wigner's semi-circle law

$$\bar{\rho}(x) = \frac{1}{\pi}\sqrt{2N - x^2}. \qquad (1.22)$$

II. ORTHOGONAL ENSEMBLE

For $\beta = 1$, we treat the following three cases separately:

$$\mathbf{A} : \{m\} = (m_1, \ldots, m_n, -m_1, \ldots, -m_n),$$
$$\mathbf{B} : \{m\} = (m_1, \ldots, m_n, -m_1, \ldots, -m_n, 0), \text{ even } N,$$
$$\mathbf{C} : \{m\} = (m_1, \ldots, m_n, -m_1, \ldots, -m_n, 0), \text{ odd } N.$$

A. even N_f

We first consider the case with $N_f \equiv 2n$ flavors and $\{m\} = (m_1, \ldots, m_n, -m_1, \ldots, -m_n)$.

We express the partition function (1.12) of the RME in terms of eigenvalues $\{x_j\}$ of H (up to a constant independent of m):

$$Z(\{m\}) = \frac{1}{N!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} dx_j \prod_{j=1}^{N} \left(e^{-x_j^2/2} \prod_{k=1}^{n}(x_j^2 + m_k^2)\right) \prod_{j>k}^{N} |x_j - x_k|. \qquad (2.1)$$

The p-level correlation function of the matrix H is defined as

$$\rho(x_1, \ldots, x_p; \{m\}) = \langle \prod_{j=1}^{p} \operatorname{tr} \delta(x_j - H) \rangle$$

$$= \frac{\Xi_p(x_1, \ldots, x_p; \{m\})}{\Xi_0(\{m\})}, \tag{2.2}$$

$$\Xi_p(x_1, \ldots, x_p; \{m\}) = \frac{1}{(N-p)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=p+1}^{N} dx_j \prod_{j=1}^{N} \left(e^{-x_j^2/2} \prod_{k=1}^{n} (x_j^2 + m_k^2) \right) \prod_{j>k}^{N} |x_j - x_k| \tag{2.3}$$

$(\Xi_0 = Z)$. We define new variables z_j as

$$\begin{aligned}
z_{2j-1} &= im_j && (j = 1, \ldots, n), \\
z_{2j} &= -im_j && (j = 1, \ldots, n), \\
z_{2n+j} &= x_j && (j = 1, \ldots, p).
\end{aligned} \tag{2.4}$$

Then the multiple integral (2.3) is expressed as

$$\Xi_p(z_1, \ldots, z_{2n+p}) = \frac{1}{\prod_{j=1}^{2n} \sqrt{w(z_j)} \prod_{j>k}^{2n} (z_j - z_k)}$$

$$\times \frac{1}{(N-p)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=2n+p+1}^{2n+N} dz_j \prod_{j=1}^{2n+N} \sqrt{w(z_j)} \prod_{j>k}^{2n+N} (z_j - z_k) \prod_{j>k>2n}^{2n+N} \operatorname{sgn}(z_j - z_k), \tag{2.5}$$

where $w(z) = e^{-z^2}$. Eq.(2.5) resembles an $(2n + p)$-level correlation function of the conventional Gaussian ensemble with $2n + N$ levels. However, conventionally the levels $z_1, \ldots, z_{2n+p}$ are all real, while in the present case some of them $(z_1, \ldots, z_{2n})$ are pure imaginary. We carefully incorporate this fact into the following evaluation.

Let us denote the integrand in Eq.(2.5) as

$$p(z_1, \ldots, z_{2n+N}) = \prod_{j=1}^{2n+N} \sqrt{w(z_j)} \prod_{j>k}^{2n+N} (z_j - z_k) \prod_{j>k>2n}^{2n+N} \operatorname{sgn}(z_j - z_k). \tag{2.6}$$

For N even, an identity

$$\prod_{j>k}^{2n+N} \operatorname{sgn}(z_j - z_k) = \operatorname{Pf}[\operatorname{sgn}(z_k - z_j)]_{j,k=1,\ldots,2n+N} \tag{2.7}$$

holds for *real* $z_1, \ldots, z_{2n+N}$. By taking the limit $z_1 < z_2 < \ldots < z_{2n} \to -\infty$, we find another identity

$$\prod_{j>k>2n}^{2n+N} \operatorname{sgn}(z_j - z_k) = \operatorname{Pf}[F_{jk}]_{j,k=1,\ldots,2n+N}, \tag{2.8}$$

where

$$F_{jk} = \begin{cases}
\operatorname{sgn}(k-j) & (j, k = 1, \ldots, 2n) \\
1 & (j = 1, \ldots, 2n;\ k = 2n+1, \ldots, 2n+N) \\
-1 & (j = 2n+1, \ldots, 2n+N;\ k = 1, \ldots, 2n) \\
\operatorname{sgn}(z_k - z_j) & (j, k = 2n+1, \ldots, 2n+N)
\end{cases}. \tag{2.9}$$

Substitution of Eq.(2.8) into Eq.(2.6) yields

$$p(z_1, \ldots, z_{2n+N}) = \prod_{j=1}^{2n+N} \sqrt{w(z_j)} \prod_{j>k}^{2n+N} (z_j - z_k) \operatorname{Pf}[F_{jk}]_{j,k=1,\ldots,2n+N}. \tag{2.10}$$

For N odd, we similarly obtain

$$p(z_1, \ldots, z_{2n+N}) = \prod_{j=1}^{2n+N} \sqrt{w(z_j)} \prod_{j>k}^{2n+N} (z_j - z_k) \operatorname{Pf} \begin{bmatrix} [F_{jk}]_{j,k=1,\ldots,2n+N} & [g_j]_{j=1,\ldots,2n+N} \\ [-g_k]_{k=1,\ldots,2n+N} & 0 \end{bmatrix}, \tag{2.11}$$

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with $g_j = g_k = 1$ $(j, k = 1, \ldots, 2n + N)$. The Pfaffians in the above can be represented as quaternion determinants [23,24,39–44]. In doing so, we introduce monic skew-orthogonal polynomials $R_j(z) = z^j + \cdots$, which satisfy the skew-orthogonality relation:

$$\langle R_{2j}, R_{2k+1} \rangle_R = -\langle R_{2k+1}, R_{2j} \rangle_R = r_j \delta_{jk},$$
$$\langle R_{2j}, R_{2k} \rangle_R = \langle R_{2j+1}, R_{2k+1} \rangle_R = 0, \tag{2.12}$$

where

$$\langle f, g \rangle_R = \int_{-\infty}^{\infty} dz \sqrt{w(z)} g(z) \int_{-\infty}^{z} dz' \sqrt{w(z')} f(z') - (f \leftrightarrow g). \tag{2.13}$$

Explicit forms for the skew-orthogonal polynomials and their norms associated with the Gaussian weight $w(z)$ are known [40]:

$$R_{2j}(z) = \frac{1}{2^{2j}} H_{2j}(z),$$
$$R_{2j+1}(z) = \frac{1}{2^{2j+1}} \Big(H_{2j+1}(z) - H'_{2j}(z) \Big), \tag{2.14}$$
$$r_j = 2^{-2j+1}(2j)! \sqrt{\pi},$$

in terms of the Hermite polynomials

$$H_j(z) = (-1)^j e^{z^2} \frac{d^j}{dz^j} e^{-z^2}. \tag{2.15}$$

Now we present the following theorems:

Theorem 1

For even N, we can rewrite $p(z_1, \ldots, z_{2n+N})$ as

$$p(z_1, \ldots, z_{2n+N}) = \Big(\prod_{j=0}^{n+N/2-1} r_j \Big) \mathrm{Tdet}[f_{jk}(z_j, z_k)]_{j,k=1,\ldots,2n+N}. \tag{2.16}$$

The quaternion elements $f_{jk}(z_j, z_k)$ are represented as

$$f_{jk}(z_j, z_k) = \begin{bmatrix} S(z_j, z_k) & I(z_j, z_k) \\ D(z_j, z_k) & S(z_k, z_j) \end{bmatrix}. \tag{2.17}$$

The functions $S(z_j, z_k)$, $D(z_j, z_k)$ and $I(z_j, z_k)$ are given by

$$S(z_j, z_k) = \sum_{\ell=0}^{n+N/2-1} \frac{\Phi_{2\ell}(z_j)\Psi_{2\ell+1}(z_k) - \Phi_{2\ell+1}(z_j)\Psi_{2\ell}(z_k)}{r_\ell},$$
$$D(z_j, z_k) = \sum_{\ell=0}^{n+N/2-1} \frac{\Psi_{2\ell}(z_j)\Psi_{2\ell+1}(z_k) - \Psi_{2\ell+1}(z_j)\Psi_{2\ell}(z_k)}{r_\ell}, \tag{2.18}$$
$$I(z_j, z_k) = -\sum_{\ell=0}^{n+N/2-1} \frac{\Phi_{2\ell}(z_j)\Phi_{2\ell+1}(z_k) - \Phi_{2\ell+1}(z_j)\Phi_{2\ell}(z_k)}{r_\ell} + F_{jk},$$

where

$$\Psi_j(z) = \sqrt{w(z)} R_j(z),$$
$$\Phi_j(z_k) = \begin{cases} \int_{-\infty}^{\infty} dz \sqrt{w(z)} R_j(z) \, \mathrm{sgn}(z_k - z) & (k = 2n+1, \ldots, 2n+N) \\ -\int_{-\infty}^{\infty} dz \sqrt{w(z)} R_j(z) & (k : \text{otherwise}) \end{cases}. \tag{2.19}$$

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Theorem 2

For odd N, we have

$$p(z_1, \ldots, z_{2n+N}) = \left(\prod_{j=0}^{n+[N/2]-1} r_j \right) s_{2n+N-1} \operatorname{Tdet}[f_{jk}^{\mathrm{odd}}(z_j, z_k)]_{j,k=1,\ldots,2n+N}. \tag{2.20}$$

The quaternion elements are represented as

$$f_{jk}^{\mathrm{odd}}(z_j, z_k) = \begin{bmatrix} S^{\mathrm{odd}}(z_j, z_k) & I^{\mathrm{odd}}(z_j, z_k) \\ D^{\mathrm{odd}}(z_j, z_k) & S^{\mathrm{odd}}(z_k, z_j) \end{bmatrix}. \tag{2.21}$$

and

$$s_j = \int_{-\infty}^{\infty} \Psi_j(z) dz. \tag{2.22}$$

The functions S^{odd}, D^{odd} and I^{odd} are given in terms of S, D and I defined in Eq.(2.18), according to

$$\begin{aligned} S^{\mathrm{odd}}(z_j, z_k) &= S(z_j, z_k)\Big|_* + \frac{\Psi_{2n+N-1}(z_k)}{s_{2n+N-1}}, \\ D^{\mathrm{odd}}(z_j, z_k) &= D(z_j, z_k)\Big|_*, \\ I^{\mathrm{odd}}(z_j, z_k) &= I(z_j, z_k)\Big|_* + \frac{\Phi_{2n+N-1}(z_j) - \Phi_{2n+N-1}(z_k)}{s_{2n+N-1}}. \end{aligned} \tag{2.23}$$

Here $*$ stands for a set of substitutions

$$R_j(z) \mapsto R_j(z) - \frac{s_j}{s_{2n+N-1}} R_{2n+N-1}(z) \qquad (j = 0, \ldots, 2n + N - 2), \tag{2.24}$$

associated with a change in the upper limit of the sum

$$n + \frac{N}{2} - 1 \mapsto n + \left[\frac{N}{2}\right] - 1. \tag{2.25}$$

Theorem 3

Let the quaternion elements q_{jk} of a selfdual $n \times n$ matrix Q_n depend on n real or complex variables $z_1, \cdots, z_n$ as

$$q_{jk} = f_{jk}(z_j, z_k). \tag{2.26}$$

We assume that $f_{jk}(z_j, z_k)$ satisfies the following conditions.

$$\int f_{nn}(z_n, z_n) d\mu(z_n) = c_n, \tag{2.27a}$$

$$\int f_{jn}(z_j, z_n) f_{nk}(z_n, z_k) d\mu(z_n) = f_{jk}(z_j, z_k) + \lambda f_{jk}(z_j, z_k) - f_{jk}(z_j, z_k)\lambda. \tag{2.27b}$$

Here $d\mu(z)$ is a suitable measure, c_n is a constant scalar, and λ is a constant quaternion. Then we have

$$\int \operatorname{Tdet} Q_n \, d\mu(z_n) = (c_n - n + 1)\operatorname{Tdet} Q_{n-1}, \tag{2.28}$$

where Q_{n-1} is the $(n-1) \times (n-1)$ matrix obtained by removing the row and the column which contain z_n.

It is straightforward to show that the quaternion elements $f_{jk}(z_j, z_k)$ and $f_{jk}^{\mathrm{odd}}(z_j, z_k)$ in *Theorem 1* and *Theorem 2*, respectively, both satisfy the conditions imposed on $f_{jk}(z_j, z_k)$ in *Theorem 3*. This means that we can inductively write

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$$\Xi_p(z_1,\ldots,z_{2n+p}) = \frac{\prod_{j=0}^{p+[N/2]-1} r_j}{\prod_{j=1}^{2n}\sqrt{w(z_j)}\prod_{j>k}^{2n}(z_j-z_k)} \times \left\{ \begin{array}{ll} \mathrm{Tdet}[f_{jk}(z_j,z_k)]_{j,k=1,\ldots,2n+p} & (N:\text{even}) \\ s_{2n+N-1}\,\mathrm{Tdet}[f_{jk}^{\text{odd}}(z_j,z_k)]_{j,k=1,\ldots,2n+p} & (N:\text{odd}) \end{array} \right. . \quad (2.29)$$

Since the final result in the asymptotic limit $N \to \infty$ should be insensitive to the parity of N, we consider only even N henceforth. Then the p-level correlation function (2.2) is written as

$$\rho(x_1,\ldots,x_p;\{m\}) = \frac{\Xi_p(z_1,\ldots,z_{2n+p})}{\Xi_0(z_1,\ldots,z_{2n})} = \frac{\mathrm{Tdet}[f_{jk}(z_j,z_k)]_{j,k=1,\ldots,2n+p}}{\mathrm{Tdet}[f_{jk}(z_j,z_k)]_{j,k=1,\ldots,2n}}. \quad (2.30)$$

We introduce a set of notations

$$\begin{aligned}
S_{jk}^{II} &= S(z_j,z_k) & (j,k=1,\ldots,2n), \\
S_{jk}^{IR} &= S(z_j,z_{2n+k}) & (j=1,\ldots,2n;\ k=1,\ldots,p), \\
S_{jk}^{RI} &= S(z_{2n+j},z_k) & (j=1,\ldots,p;\ k=1,\ldots,2n), \\
S_{jk}^{RR} &= S(z_{2n+j},z_{2n+k}) & (j,k=1,\ldots,p)
\end{aligned} \quad (2.31)$$

and similarly for D and I. Using Dyson's identity (A.6) we can rewrite the correlation functions as

$$\rho(x_1,\ldots,x_p;\{m\}) = (-1)^{p(p-1)/2}\frac{\mathrm{Pf}\begin{bmatrix} -I^{II} & S^{II} & -I^{IR} & S^{IR} \\ -(S^{II})^T & D^{II} & -(S^{RI})^T & D^{IR} \\ -I^{RI} & S^{RI} & -I^{RR} & S^{RR} \\ -(S^{IR})^T & D^{RI} & -(S^{RR})^T & D^{RR} \end{bmatrix}}{\mathrm{Pf}\begin{bmatrix} -I^{II} & S^{II} \\ -(S^{II})^T & D^{II} \end{bmatrix}}$$

$$= (-1)^{p(p-1)/2}\frac{\mathrm{Pf}\begin{bmatrix} D^{II} & -(S^{RI})^T & -(D^{RI})^T \\ S^{RI} & -I^{RR} & S^{RR} \\ D^{RI} & -(S^{RR})^T & D^{RR} \end{bmatrix}}{\mathrm{Pf}\,[D^{II}]}. \quad (2.32)$$

In the last line we have exploited a Pfaffian identity that holds for antisymmetric matrices A, B of even ranks and a row vector v:

$$\mathrm{Pf}\begin{bmatrix} & & & v \\ & A & & \vdots \\ & & & v \\ \hline -v^T & \cdots & -v^T & B \end{bmatrix} = \mathrm{Pf}[A]\,\mathrm{Pf}[B]. \quad (2.33)$$

Now we proceed to evaluate the component functions of the quaternion kernel $f_{jk}(z_j,z_k)$ in the asymptotic limit (1.13) and (1.20), where unfolded microscopic variables

$$\begin{aligned}
\sqrt{2N}z_{2j-1} &\equiv \zeta_{2j-1} = i\mu_j & (j=1,\ldots,n), \\
\sqrt{2N}z_{2j} &\equiv \zeta_{2j} = -i\mu_j & (j=1,\ldots,n), \\
\sqrt{2N}z_{2n+j} &\equiv \lambda_j & (j=1,\ldots,p)
\end{aligned} \quad (2.34)$$

are kept fixed. We note that all elements of the sub-matrices that appear in the second line of Eq.(2.32) are expressed as (derivatives or integrals of) an analytic function

$$\begin{aligned}
\bar{S}(z,z') &= \sum_{j=0}^{n+N/2-1}\frac{\bar{\Phi}_{2j}(z)\Psi_{2j+1}(z')-\bar{\Phi}_{2j+1}(z)\Psi_{2j}(z')}{r_j} \\
&= \frac{e^{-(z^2+z'^2)/2}}{2^{2n+N}\sqrt{\pi}\Gamma(2n+N)}\frac{H_{2n+N}(z)H_{2n+N-1}(z')-H_{2n+N-1}(z)H_{2n+N}(z')}{z-z'} \\
&\quad + \frac{e^{-z'^2/2}}{2^{2n+N}\sqrt{\pi}\Gamma(2n+N)}H_{2n+N-1}(z')\int_0^z e^{-u^2/2}H_{2n+N}(u)du,
\end{aligned} \quad (2.35)$$

where

$$\bar{\Phi}_j(z) = \left\{ \int_{-\infty}^{z} - \int_{z}^{\infty} \right\} \Psi_j(u)du, \tag{2.36}$$

with real and/or imaginary arguments:

$$D_{jk}^{II} = \frac{1}{2}\frac{\partial}{\partial z_j}\bar{S}(z_j, z_k), \tag{2.37a}$$

$$S_{jk}^{RI} = \bar{S}(x_j, z_k), \tag{2.37b}$$

$$D_{jk}^{RI} = \frac{1}{2}\frac{\partial}{\partial x_j}\bar{S}(x_j, z_k), \tag{2.37c}$$

$$S_{jk}^{RR} = \bar{S}(x_j, x_k), \tag{2.37d}$$

$$D_{jk}^{RR} = \frac{1}{2}\frac{\partial}{\partial x_j}\bar{S}(x_j, x_k), \tag{2.37e}$$

$$I_{jk}^{RR} = -2\int_{x_j}^{x_k}\bar{S}(x_j, x)dx - \mathrm{sgn}(x_j - x_k). \tag{2.37f}$$

In the second line of Eq.(2.35), we have singled out the unitary scalar kernel and applied to it the Christoffel-Darboux formula. Substituting asymptotic formulas for the Hermite polynomials

$$H_{2k}(z) \sim \frac{(-1)^k 2^{2k} k!}{\sqrt{\pi k}}\cos(2\sqrt{k}z),$$

$$H_{2k+1}(z) \sim \frac{(-1)^k 2^{2k+1} k!}{\sqrt{\pi}}\sin(2\sqrt{k}z), \tag{2.38}$$

valid under $k \to \infty$, $z \to 0$, $\sqrt{k}z$: fixed, one can show that $\bar{S}(z, z')$ approaches the sine kernel [40]:

$$\frac{1}{\sqrt{2N}}\bar{S}(\frac{\zeta}{\sqrt{2N}}, \frac{\zeta'}{\sqrt{2N}}) \sim \frac{\sin(\zeta - \zeta')}{\pi(\zeta - \zeta')} \equiv K(\zeta - \zeta'). \tag{2.39}$$

The Pfaffian elements in the asymptotic limit, after taking into account an unfolding by the factor $\sqrt{2N}$, are then expressed in terms of $K(\zeta)$:

$$\mathbf{D}_{jk}^{II} \equiv \frac{1}{2N}D(z_j, z_k) \sim \frac{1}{2}K'(\zeta_j - \zeta_k), \tag{2.40a}$$

$$\mathbf{S}_{jk}^{RI} \equiv \frac{1}{\sqrt{2N}}S(x_j, z_k) \sim K(\lambda_j - \zeta_k), \tag{2.40b}$$

$$\mathbf{D}_{jk}^{RI} \equiv \frac{1}{2N}D(x_j, z_k) \sim \frac{1}{2}K'(\lambda_j - \zeta_k), \tag{2.40c}$$

$$\mathbf{S}_{jk}^{RR} \equiv \frac{1}{\sqrt{2N}}S(x_j, x_k) \sim K(\lambda_j - \lambda_k), \tag{2.40d}$$

$$\mathbf{D}_{jk}^{RR} \equiv \frac{1}{2N}D(x_j, x_k) \sim \frac{1}{2}K'(\lambda_j - \lambda_k), \tag{2.40e}$$

$$\mathbf{I}_{jk}^{RR} \equiv I(x_j, x_k) \sim 2\int_{0}^{\lambda_j - \lambda_k}K(\lambda)d\lambda - \mathrm{sgn}(\lambda_j - \lambda_k). \tag{2.40f}$$

These matrix elements constitute the finite-volume partition function

$$\mathcal{Z}(\{\mu\}) \equiv \Xi_0(\{\frac{\mu}{\sqrt{2N}}\})$$

$$= \mathrm{const.}\frac{\mathrm{Pf}[\mathbf{D}^{II}]}{\prod_{j=1}^{n}\mu_j \prod_{j>k}^{n}(\mu_j^2 - \mu_k^2)^2}, \tag{2.41}$$

and the scaled spectral correlation functions

$$\rho_s(\lambda_1,\ldots,\lambda_p;\{\mu\}) \equiv \left(\frac{1}{\sqrt{2N}}\right)^p \rho(\frac{\lambda_1}{\sqrt{2N}},\ldots,\frac{\lambda_p}{\sqrt{2N}};\{\frac{\mu}{\sqrt{2N}}\})$$

$$= (-1)^{p(p-1)/2} \frac{\mathrm{Pf} \begin{bmatrix} \mathbf{D}^{II} & -(\mathbf{S}^{RI})^T & -(\mathbf{D}^{RI})^T \\ \mathbf{S}^{RI} & -\mathbf{I}^{RR} & \mathbf{S}^{RR} \\ \mathbf{D}^{RI} & -(\mathbf{S}^{RR})^T & \mathbf{D}^{RR} \end{bmatrix}}{\mathrm{Pf}\,[\mathbf{D}^{II}]}. \tag{2.42}$$

In the quenched limit $\mu_1,\ldots,\mu_n \to \infty$ when the ratio of two Pfaffians is replaced by a minor $\mathrm{Pf}\begin{bmatrix} -\mathbf{I}^{RR} & \mathbf{S}^{RR} \\ -(\mathbf{S}^{RR})^T & \mathbf{D}^{RR} \end{bmatrix}$, the correlation functions approach those of the Gaussian orthogonal ensemble [40]. By the same token, it satisfies a sequence

$$\rho_s(\{\lambda\};\mu_1,\ldots,\mu_n,-\mu_1,\ldots,-\mu_n) \stackrel{\mu_n\to\infty}{\longrightarrow} \rho_s(\{\lambda\};\mu_1,\ldots,\mu_{n-1},-\mu_1,\ldots,-\mu_{n-1}) \stackrel{\mu_{n-1}\to\infty}{\longrightarrow} \cdots, \tag{2.43}$$

as each of the masses is decoupled by going to infinity. To illustrate this decoupling, we exhibit in FIG.1 a plot of the spectral density $\rho_s(\lambda;\mu,-\mu)$ $(p=1,n=1)$.

B. odd N_f, even N

Next we consider the case with $N_f \equiv 2n+1$ flavors, $\{m\} = (m_1,\ldots,m_n,-m_1,\ldots,-m_n,0)$, and with even N. We express the partition function (1.17) of the RME in terms of eigenvalues $\{x_j\}$ of H:

$$Z(\{m\}) = \frac{1}{N!} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{j=1}^{N} dx_j \prod_{j=1}^{N}\left(e^{-x_j^2/2}x_j \prod_{k=1}^{n}(x_j^2+m_k^2)\right) \prod_{j>k}^{N} |x_j - x_k|. \tag{2.44}$$

The p-level correlation function of the matrix H is defined as

$$\rho(x_1,\ldots,x_p;\{m\}) = \langle \prod_{j=1}^{p} \mathrm{tr}\,\delta(x_j - H)\rangle$$

$$= \frac{\Xi_p(x_1,\ldots,x_p;\{m\})}{\Xi_0(\{m\})}, \tag{2.45}$$

$$\Xi_p(x_1,\ldots,x_p;\{m\}) = \frac{1}{(N-p)!} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{j=p+1}^{N} dx_j \prod_{j=1}^{N}\left(e^{-x_j^2/2}x_j \prod_{k=1}^{n}(x_j^2+m_k^2)\right) \prod_{j>k}^{N} |x_j - x_k| \tag{2.46}$$

$(\Xi_0 = Z)$. We define new variables z_j as

$$\begin{aligned} z_0 &= 0, \\ z_{2j-1} &= im_j \quad (j=1,\ldots,n), \\ z_{2j} &= -im_j \quad (j=1,\ldots,n), \\ z_{2n+j} &= x_j \quad (j=1,\ldots,p). \end{aligned} \tag{2.47}$$

Then the multiple integral (2.46) is expressed as

$$\Xi_p(z_0,\ldots,z_{2n+p}) = \frac{1}{\prod_{j=0}^{2n}\sqrt{w(z_j)}\prod_{j>k\geq0}^{2n}(z_j - z_k)}$$

$$\times \frac{1}{(N-p)!} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{j=2n+p+1}^{2n+N} dz_j \prod_{j=0}^{2n+N}\sqrt{w(z_j)} \prod_{j>k\geq0}^{2n+N}(z_j - z_k) \prod_{j>k>2n}^{2n+N} \mathrm{sgn}(z_j - z_k), \tag{2.48}$$

where $w(z) = e^{-z^2}$.

Let us denote the integrand in Eq.(2.48) as

$$p(z_0,\ldots,z_{2n+N}) = \prod_{j=0}^{2n+N}\sqrt{w(z_j)} \prod_{j>k\geq0}^{2n+N}(z_j - z_k) \prod_{j>k>2n}^{2n+N} \mathrm{sgn}(z_j - z_k). \tag{2.49}$$

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An identity

$$\prod_{\substack{j>k\geq 0}}^{2n+N+1} \mathrm{sgn}(z_j - z_k) = \mathrm{Pf}[\mathrm{sgn}(z_k - z_j)]_{j,k=0,\ldots,2n+N+1} \tag{2.50}$$

holds for *real* $z_0, \ldots, z_{2n+N+1}$. By taking the limit $z_0 < z_1 < \ldots < z_{2n} \to -\infty$ and $z_{2n+N+1} \to +\infty$, we find another identity

$$\prod_{\substack{j>k>2n}}^{2n+N+1} \mathrm{sgn}(z_j - z_k) = \mathrm{Pf} \left[\begin{array}{cc} [F_{jk}]_{j,k=0,\ldots,2n+N} & [g_j]_{j=0,\ldots,2n+N} \\ [-g_k]_{k=0,\ldots,2n+N} & 0 \end{array} \right], \tag{2.51}$$

where

$$F_{jk} = \begin{cases} \mathrm{sgn}(k-j) & (j,k = 0, \ldots, 2n) \\ 1 & (j = 0, \ldots, 2n; \ k = 2n+1, \ldots, 2n+N) \\ -1 & (j = 2n+1, \ldots, 2n+N; \ k = 0, \ldots, 2n) \\ \mathrm{sgn}(z_k - z_j) & (j,k = 2n+1, \ldots, 2n+N) \end{cases} \tag{2.52}$$

and $g_j = g_k = 1$ $(j,k = 0, \ldots, 2n+N)$. Substitution of Eq.(2.51) into Eq.(2.49) yields

$$p(z_0, \ldots, z_{2n+N}) = \prod_{j=0}^{2n+N} \sqrt{w(z_j)} \prod_{j>k\geq 0}^{2n+N} (z_j - z_k) \mathrm{Pf} \left[\begin{array}{cc} [F_{jk}]_{j,k=0,\ldots,2n+N} & [g_j]_{j=0,\ldots,2n+N} \\ [-g_k]_{k=0,\ldots,2n+N} & 0 \end{array} \right]. \tag{2.53}$$

The Pfaffian in the above can be represented as a quaternion determinant, due to the following theorem (which is essentially *Theorem 2*):

Theorem 2'

For even N, we can rewrite $p(z_0, \ldots, z_{2n+N})$ as

$$p(z_0, \ldots, z_{2n+N}) = \left(\prod_{j=0}^{n+N/2-1} r_j \right) s_{2n+N} \, \mathrm{Tdet}[f_{jk}^{\mathrm{even}}(z_j, z_k)]_{j,k=0,\ldots,2n+N}. \tag{2.54}$$

The quaternion elements are represented as

$$f_{jk}^{\mathrm{even}}(z_j, z_k) = \left[\begin{array}{cc} S^{\mathrm{even}}(z_j, z_k) & I^{\mathrm{even}}(z_j, z_k) \\ D^{\mathrm{even}}(z_j, z_k) & S^{\mathrm{even}}(z_k, z_j) \end{array} \right]. \tag{2.55}$$

and s_j is defined in Eq.(2.22). The functions S^{even}, D^{even} and I^{even} are given in terms of S, D and I defined in Eq.(2.18), according to

$$\begin{aligned} S^{\mathrm{even}}(z_j, z_k) &= S(z_j, z_k)\Big|_{\#} + \frac{\Psi_{2n+N}(z_k)}{s_{2n+N}}, \\ D^{\mathrm{even}}(z_j, z_k) &= D(z_j, z_k)\Big|_{\#}, \\ I^{\mathrm{even}}(z_j, z_k) &= I(z_j, z_k)\Big|_{\#} + \frac{\Phi_{2n+N}(z_j) - \Phi_{2n+N}(z_k)}{s_{2n+N}}. \end{aligned} \tag{2.56}$$

Here $\#$ stands for a set of substitutions

$$R_j(z) \mapsto R_j(z) - \frac{s_j}{s_{2n+N}} R_{2n+N}(z) \qquad (j = 0, \ldots, 2n+N-1). \tag{2.57}$$

It is straightforward to show that the quaternion element $f_{jk}^{\mathrm{even}}(z_j, z_k)$ in *Theorem 2'* satisfies the conditions imposed on $f_{jk}(z_j, z_k)$ in *Theorem 3*. This means that we can inductively write

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$$\Xi_p(z_0,\ldots,z_{2n+p}) = \frac{(\prod_{j=0}^{p+N/2-1} r_j)s_{2n+N}}{\prod_{j=0}^{2n}\sqrt{w(z_j)}\prod_{j>k\geq0}^{2n}(z_j - z_k)}\mathrm{Tdet}[f_{jk}^{\mathrm{even}}(z_j,z_k)]_{j,k=0,\ldots,2n+p}. \tag{2.58}$$

Then the p-level correlation function (2.45) is written as

$$\rho(x_1,\ldots,x_p;\{m\}) = \frac{\Xi_p(z_0,\ldots,z_{2n+p})}{\Xi_0(z_0,\ldots,z_{2n})} = \frac{\mathrm{Tdet}[f_{jk}^{\mathrm{even}}(z_j,z_k)]_{j,k=0,\ldots,2n+p}}{\mathrm{Tdet}[f_{jk}^{\mathrm{even}}(z_j,z_k)]_{j,k=0,\ldots,2n}}. \tag{2.59}$$

We introduce a set of notations

$$\begin{aligned}
S_{jk}^{II} &= S^{\mathrm{even}}(z_j,z_k) & (j,k=0,\ldots,2n),\\
S_{jk}^{IR} &= S^{\mathrm{even}}(z_j,z_{2n+k}) & (j=0,\ldots,2n;\ k=1,\ldots,p),\\
S_{jk}^{RI} &= S^{\mathrm{even}}(z_{2n+j},z_k) & (j=1,\ldots,p;\ k=0,\ldots,2n),\\
S_{jk}^{RR} &= S^{\mathrm{even}}(z_{2n+j},z_{2n+k}) & (j,k=1,\ldots,p)
\end{aligned} \tag{2.60}$$

and similarly for D and I. A simplification

$$\begin{aligned}
S_{jk}^{II} &= S^{\mathrm{even}}(-\infty,z_k) = \frac{\Psi_{2n+N}(z_k)}{s_{2n+N}} \equiv S_k^I,\\
S_{jk}^{IR} &= S^{\mathrm{even}}(-\infty,z_{2n+k}) = \frac{\Psi_{2n+N}(z_{2n+k})}{s_{2n+N}} \equiv S_k^R,\\
I_{jk}^{IR} &= -I_{kj}^{RI} = I^{\mathrm{even}}(-\infty,z_{2n+k}) = -\frac{\Phi_{2n+N}(z_{2n+k})}{s_{2n+N}} \equiv I_k^R,
\end{aligned} \tag{2.61}$$

results from the definitions (2.18) and (2.56). Using Dyson's identity (A.6) we can rewrite the correlation functions as

$$\begin{aligned}
\rho(x_1,\ldots,x_p;\{m\}) &= (-1)^{p(p-1)/2}\frac{\mathrm{Pf}\begin{bmatrix} -I^{II} & S^{II} & -I^{IR} & S^{IR}\\ -(S^{II})^T & D^{II} & -(S^{RI})^T & D^{IR}\\ -I^{RI} & S^{RI} & -I^{RR} & S^{RR}\\ -(S^{IR})^T & D^{RI} & -(S^{RR})^T & D^{RR}\end{bmatrix}}{\mathrm{Pf}\begin{bmatrix} -I^{II} & S^{II}\\ -(S^{II})^T & D^{II}\end{bmatrix}}\\[2em]
&= (-1)^{p(p-1)/2}\frac{\mathrm{Pf}\begin{bmatrix} 0 & S^I & -I^R & S^R\\ -(S^I)^T & D^{II} & -(S^{RI})^T & -(D^{RI})^T\\ (I^R)^T & S^{RI} & -I^{RR} & S^{RR}\\ -(S^R)^T & D^{RI} & -(S^{RR})^T & D^{RR}\end{bmatrix}}{\mathrm{Pf}\begin{bmatrix} 0 & S^I\\ -(S^I)^T & D^{II}\end{bmatrix}}.
\end{aligned} \tag{2.62}$$

In the last line we have exploited a Pfaffian identity that holds for antisymmetric matrices A, B of odd ranks and a row vector v:

$$\mathrm{Pf}\left[\begin{array}{c|c} A & \begin{matrix}v\\\vdots\\v\end{matrix}\\ \hline -v^T\cdots-v^T & B\end{array}\right] = \mathrm{Pf}\left[\begin{array}{c|c} A & \begin{matrix}1\\\vdots\\1\end{matrix}\\ \hline -1\cdots-1 & 0\end{array}\right]\mathrm{Pf}\begin{bmatrix} 0 & v\\ -v^T & B\end{bmatrix}. \tag{2.63}$$

Likewise previous Subsection, we proceed to evaluate the component functions of the quaternion kernel $f_{jk}^{\mathrm{even}}(z_j,z_k)$ in the asymptitic limit where unfolded microscopic variables

$$\begin{aligned}
\sqrt{2N}z_0 &\equiv \zeta_0 = 0,\\
\sqrt{2N}z_{2j-1} &\equiv \zeta_{2j-1} = i\mu_j & (j=1,\ldots,n),\\
\sqrt{2N}z_{2j} &\equiv \zeta_{2j} = -i\mu_j & (j=1,\ldots,n),\\
\sqrt{2N}z_{2n+j} &\equiv \lambda_j & (j=1,\ldots,p),
\end{aligned} \tag{2.64}$$

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are kept fixed. We note that all elements of the sub-matrices that appear in the second line of Eq.(2.62) are expressed as (derivatives or integrals of) analytic functions $\bar{S}(z, z')$ and $\bar{\Phi}_j(z)$ defined in Eqs.(2.35) and (2.36), and $\Psi_j(z)$:

$$D_{jk}^{II} = \frac{1}{2} \frac{\partial}{\partial z_j} \bar{S}(z_j, z_k), \tag{2.65a}$$

$$S_{jk}^{RI} = \bar{S}(x_j, z_k) + \frac{\bar{\bar{\Phi}}_{2\alpha+N}(x_j)}{s_{2\alpha+N}} \bar{S}(-\infty, z_k) - \frac{\Psi_{2\alpha+N}(z_k)}{s_{2\alpha+N}} \left(2 \int_{-\infty}^{x_j} \bar{S}(-\infty, x)dx - 1 \right), \tag{2.65b}$$

$$D_{jk}^{RI} = \frac{1}{2} \frac{\partial}{\partial x_j} \bar{S}(x_j, z_k), \tag{2.65c}$$

$$S_{jk}^{RR} = \bar{S}(x_j, x_k) + \frac{\bar{\bar{\Phi}}_{2\alpha+N}(x_j)}{s_{2\alpha+N}} \bar{S}(-\infty, x_k) - \frac{\Psi_{2\alpha+N}(x_k)}{s_{2\alpha+N}} \left(2 \int_{-\infty}^{x_j} \bar{S}(-\infty, x)dx - 1 \right), \tag{2.65d}$$

$$D_{jk}^{RR} = \frac{1}{2} \frac{\partial}{\partial x_j} \bar{S}(x_j, x_k), \tag{2.65e}$$

$$I_{jk}^{RR} = -2 \int_{x_j}^{x_k} \bar{S}(x_j, x)dx - \text{sgn}(x_j - x_k)$$
$$+ \frac{\bar{\bar{\Phi}}_{2\alpha+N}(x_k)}{s_{2\alpha+N}} \left(2 \int_{-\infty}^{x_j} \bar{S}(-\infty, x)dx - 1 \right) - \frac{\bar{\bar{\Phi}}_{2\alpha+N}(x_j)}{s_{2\alpha+N}} \left(2 \int_{-\infty}^{x_k} \bar{S}(-\infty, x)dx - 1 \right). \tag{2.65f}$$

Utilizing an identity

$$\int_0^\infty e^{-u^2/2} H_{2k}(u)du = 2^{2k-1/2}\Gamma(k + \frac{1}{2}), \tag{2.66}$$

an asymptotic formula

$$\int_0^\infty e^{-u^2/2} H_{2k-1}(u)du = \frac{2^{2k-1}\Gamma(k)}{\sqrt{\pi}} \sum_{\ell=0}^{k-1} (-1)^\ell \frac{\Gamma(\ell + 1/2)}{\Gamma(\ell + 1)}$$
$$= \frac{2^{2k-1}\Gamma(k)}{\sqrt{\pi}} \left(\sqrt{\pi} \, {}_2F_1(1, \frac{1}{2}; 1; -1) - \sum_{\ell=k}^{\infty} (-1)^\ell \frac{\Gamma(\ell + 1/2)}{\Gamma(\ell + 1)} \right)$$
$$\sim \frac{2^{2k-1}\Gamma(k)}{\sqrt{\pi}} \left(\sqrt{\frac{\pi}{2}} - \frac{(-1)^k}{2\sqrt{k}} \right) \qquad (k \gg 1), \tag{2.67}$$

and Eq.(2.38), we obtain Eqs.(2.40) (with S, D, I in the second places replaced by S^{even}, D^{even}, I^{even}) and

$$\mathbf{S}_j^I \equiv S^I(z_j) \sim \frac{(-1)^{n+N/2}}{\sqrt{2\pi}} \cos \zeta_j, \tag{2.68a}$$

$$\mathbf{S}_j^R \equiv S^R(x_j) \sim \frac{(-1)^{n+N/2}}{\sqrt{2\pi}} \cos \lambda_j, \tag{2.68b}$$

$$\mathbf{I}_j^R \equiv \sqrt{2N} I^R(x_j) \sim (-1)^{n+N/2+1} \sqrt{\frac{2}{\pi}} \sin \lambda_j. \tag{2.68c}$$

These matrix elements constitute the finite-volume partition function

$$\mathcal{Z}(\{\mu\}) \equiv \Xi_0(\{\frac{\mu}{\sqrt{2N}}\})$$
$$= \text{const.} \frac{\text{Pf} \begin{bmatrix} 0 & \mathbf{S}^I \\ -(\mathbf{S}^I)^T & \mathbf{D}^{II} \end{bmatrix}}{\prod_{j=1}^n \mu_j^3 \prod_{j>k}^n (\mu_j^2 - \mu_k^2)^2}, \tag{2.69}$$

and the scaled spectral correlation functions

$$\rho_s(\lambda_1, \ldots, \lambda_p; \{\mu\}) \equiv \left(\frac{1}{\sqrt{2N}} \right)^p \rho(\frac{\lambda_1}{\sqrt{2N}}, \ldots, \frac{\lambda_p}{\sqrt{2N}}; \{\frac{\mu}{\sqrt{2N}}\})$$

$$
\begin{aligned}
&\mathrm{Pf}\begin{bmatrix} 0 & \mathbf{S}^I & -\mathbf{I}^R & \mathbf{S}^R \\ -(\mathbf{S}^I)^T & \mathbf{D}^{II} & -(\mathbf{S}^{RI})^T & -(\mathbf{D}^{RI})^T \\ (\mathbf{I}^R)^T & \mathbf{S}^{RI} & -\mathbf{I}^{RR} & \mathbf{S}^{RR} \\ -(\mathbf{S}^R)^T & \mathbf{D}^{RI} & -(\mathbf{S}^{RR})^T & \mathbf{D}^{RR} \end{bmatrix} \\
&= (-1)^{p(p-1)/2} \frac{}{\mathrm{Pf}\begin{bmatrix} 0 & \mathbf{S}^I \\ -(\mathbf{S}^I)^T & \mathbf{D}^{II} \end{bmatrix}}.
\end{aligned}
\tag{2.70}
$$

It satisfies a sequence

$$
\rho_s(\{\lambda\}; \mu_1, \ldots, \mu_n, -\mu_1, \ldots, -\mu_n, 0) \xrightarrow{\mu_n \to \infty} \rho_s(\{\lambda\}; \mu_1, \ldots, \mu_{n-1}, -\mu_1, \ldots, -\mu_{n-1}, 0) \xrightarrow{\mu_{n-1} \to \infty} \cdots,
\tag{2.71}
$$

as each of the masses is decoupled by going to infinity. To illustrate this decoupling, we exhibit in FIG.2 a plot of the spectral density $\rho_s(\lambda; \mu, -\mu, 0)$ $(p = 1, n = 1)$.

C. odd N_f, odd N

We finally consider the case with $N_f \equiv 2n + 1$ flavors, $\{m\} = (m_1, \ldots, m_n, -m_1, \ldots, -m_n, 0)$, and with odd N. As mentioned in Introduction, this case is pathological because

$$
\Xi_p(-x_1, \ldots, -x_p; \{m\}) = -\Xi_p(x_1, \ldots, x_p; \{m\})
\tag{2.72}
$$

and namely $\Xi_0(\{m\}) = Z(\{m\}) = 0$. The quantities computed below should not be considered as correlation functions of a RME, but merely as multiple integrals defined by Eqs.(2.44)$\sim$(2.46).

As in previous Subsection, we define variables z_j by Eq.(2.47), and denote the integrand in Ξ_p as

$$
p(z_0, \ldots, z_{2n+N}) = \prod_{j=0}^{2n+N} \sqrt{w(z_j)} \prod_{j>k\geq 0}^{2n+N} (z_j - z_k) \prod_{j>k>2n}^{2n+N} \mathrm{sgn}(z_j - z_k).
\tag{2.73}
$$

An identity

$$
\prod_{j>k\geq 0}^{2n+N} \mathrm{sgn}(z_j - z_k) = \mathrm{Pf}[\mathrm{sgn}(z_k - z_j)]_{j,k=0,\ldots,2n+N}
\tag{2.74}
$$

holds for *real* $z_0, z_1, \ldots, z_{2n+N}$. By taking the limit $z_0 < z_1 < \ldots < z_{2n} \to -\infty$, we find another identity

$$
\prod_{j>k>2n}^{2n+N} \mathrm{sgn}(z_j - z_k) = \mathrm{Pf}[F_{jk}]_{j,k=0,\ldots,2n+N},
\tag{2.75}
$$

where F_{jk} is defined in Eq.(2.52). Substitution of Eq.(2.75) into Eq.(2.73) yields

$$
p(z_0, \ldots, z_{2n+N}) = \prod_{j=0}^{2n+N} \sqrt{w(z_j)} \prod_{j>k\geq 0}^{2n+N} (z_j - z_k) \mathrm{Pf}[F_{jk}]_{j,k=0,\ldots,2n+N}.
\tag{2.76}
$$

The Pfaffian in the above can be represented as a quaternion determinant, due to the following theorem (which is essentially *Theorem 1*)

Theorem 1'

For odd N, we can rewrite $p(z_0, \ldots, z_{2n+N})$ as

$$
p(z_0, \ldots, z_{2n+N}) = \left(\prod_{j=0}^{n+(N+1)/2-1} r_j \right) \mathrm{Tdet}[f_{jk}(z_j, z_k)]_{j,k=0,\ldots,2n+N}.
\tag{2.77}
$$

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The quaternion elements $f_{jk}(z_j, z_k)$ are represented as

$$f_{jk}(z_j, z_k) = \begin{bmatrix} S(z_j, z_k) & I(z_j, z_k) \\ D(z_j, z_k) & S(z_k, z_j) \end{bmatrix}. \tag{2.78}$$

The functions $S(z_j, z_k)$, $D(z_j, z_k)$ and $I(z_j, z_k)$ are given by

$$S(z_j, z_k) = \sum_{\ell=0}^{n+(N+1)/2-1} \frac{\Phi_{2\ell}(z_j)\Psi_{2\ell+1}(z_k) - \Phi_{2\ell+1}(z_j)\Psi_{2\ell}(z_k)}{r_\ell},$$

$$D(z_j, z_k) = \sum_{\ell=0}^{n+(N+1)/2-1} \frac{\Psi_{2\ell}(z_j)\Psi_{2\ell+1}(z_k) - \Psi_{2\ell+1}(z_j)\Psi_{2\ell}(z_k)}{r_\ell}, \tag{2.79}$$

$$I(z_j, z_k) = -\sum_{\ell=0}^{n+(N+1)/2-1} \frac{\Phi_{2\ell}(z_j)\Phi_{2\ell+1}(z_k) - \Phi_{2\ell+1}(z_j)\Phi_{2\ell}(z_k)}{r_\ell} + F_{jk}.$$

As before, the above quaternion elements satisfy the conditions imposed on $f_{jk}(z_j, z_k)$ in *Theorem 3*, so that we can inductively write

$$\Xi_p(z_0, \ldots, z_{2n+p}) = \frac{\prod_{j=0}^{p+(N+1)/2-1} r_j}{\prod_{j=0}^{2n} \sqrt{w(z_j)} \prod_{j>k\geq 0}^{2n}(z_j - z_k)} \text{Tdet}[f_{jk}(z_j, z_k)]_{j,k=0,\ldots,2n+p}. \tag{2.80}$$

In order to circumvent the vanishing of the partition function, we regard $z_0, z_1, \ldots, z_{2n}$ as generic variables, and define the 'p-level correlation function'

$$\rho(x_1, \cdots, x_p; \{m\}) \equiv \frac{\Xi_p(z_0, \cdots, z_{2n+p})}{\Xi_0(z_0, \cdots, z_{2n})} = \frac{\text{Tdet}[f_{jk}(z_j, z_k)]_{j,k=0,\cdots,2n+p}}{\text{Tdet}[f_{jk}(z_j, z_k)]_{j,k=0,\cdots,2n}}. \tag{2.81}$$

We introduce a set of notations

$$\begin{aligned}
S_{jk}^{II} &= S(z_j, z_k) & (j, k = 0, \ldots, 2n), \\
S_{jk}^{IR} &= S(z_j, z_{2n+k}) & (j = 0, \ldots, 2n; \ k = 1, \ldots, p), \\
S_{jk}^{RI} &= S(z_{2n+j}, z_k) & (j = 1, \ldots, p; \ k = 0, \ldots, 2n), \\
S_{jk}^{RR} &= S(z_{2n+j}, z_{2n+k}) & (j, k = 1, \ldots, p)
\end{aligned} \tag{2.82}$$

and similarly for D and I. A simplification

$$\begin{aligned}
S_{jk}^{II} &= S^{II}(-\infty, z_k) \equiv S_k^I, \\
I_{jk}^{IR} &= -I_{kj}^{RI} = I^{IR}(-\infty, z_{2n+k}) \equiv I_k^R, \\
S_{jk}^{IR} &= S^{IR}(-\infty, z_{2n+k}) \equiv S_k^R,
\end{aligned} \tag{2.83}$$

results again from the definition (2.79). Using Dyson's identity (A.6) we can rewrite the 'correlation functions' as Eq.(2.62).

Likewise two previous Subsections, we proceed to evaluate the component functions of the quaternion kernel $f_{jk}(z_j, z_k)$ in the asymptitic limit where unfolded microscopic variables (2.64) are kept fixed. We note that all elements (other than S^I, S^R, and I^R) of the sub-matrices that appear in the second line of Eq.(2.62) are expressed as (derivatives or integrals of) an analytic function $\bar{S}(z, z')$ defined in Eq.(2.35) (with N replaced by $N+1$), with real and/or imaginary arguments:

$$D_{jk}^{II} = \frac{1}{2}\frac{\partial}{\partial z_j}\bar{S}(z_j, z_k), \tag{2.84a}$$

$$S_{jk}^{RI} = \bar{S}(x_j, z_k), \tag{2.84b}$$

$$D_{jk}^{RI} = \frac{1}{2}\frac{\partial}{\partial x_j}\bar{S}(x_j, z_k), \tag{2.84c}$$

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$$S_{jk}^{RR} = \bar{S}(x_j, x_k), \tag{2.84d}$$

$$D_{jk}^{RR} = \frac{1}{2}\frac{\partial}{\partial x_j}\bar{S}(x_j, x_k), \tag{2.84e}$$

$$I_{jk}^{RR} = -2\int_{x_j}^{x_k}\bar{S}(x_j, x)dx - \text{sgn}(x_j - x_k). \tag{2.84f}$$

We obtain Eqs.(2.40) and

$$\mathbf{S}_j^I \equiv S^I(z_j) \sim \frac{(-1)^{n+(N+1)/2}}{\sqrt{2\pi}}\sin\zeta_j, \tag{2.85a}$$

$$\mathbf{S}_j^R \equiv S^R(x_j) \sim \frac{(-1)^{n+(N+1)/2}}{\sqrt{2\pi}}\sin\lambda_j, \tag{2.85b}$$

$$\mathbf{I}_j^R \equiv \sqrt{2N}I^R(x_j) \sim (-1)^{n+(N+1)/2}\sqrt{\frac{2}{\pi}}\cos\lambda_j. \tag{2.85c}$$

Note the phase shifts of the trigonometric functions between Eq.(2.85) and its even-N counterpart (2.68). These matrix elements constitute the 'finite-volume partition function'

$$\mathcal{Z}(\{\mu\}) \equiv \Xi_0(\{\frac{\mu}{\sqrt{2N}}\})$$

$$= \text{const.}\frac{\text{Pf}\begin{bmatrix} 0 & \mathbf{S}^I \\ -(\mathbf{S}^I)^T & \mathbf{D}^{II} \end{bmatrix}}{\prod_{j=1}^n \mu_j^3 \prod_{j>k}^n (\mu_j^2 - \mu_k^2)^2}, \tag{2.86}$$

and the 'scaled correlation functions'

$$\rho_s(\lambda_1, \ldots, \lambda_p; \{\mu\}) \equiv \left(\frac{1}{\sqrt{2N}}\right)^p \rho(\frac{\lambda_1}{\sqrt{2N}}, \ldots, \frac{\lambda_p}{\sqrt{2N}}; \{\frac{\mu}{\sqrt{2N}}\})$$

$$= (-1)^{p(p-1)/2}\frac{\text{Pf}\begin{bmatrix} 0 & \mathbf{S}^I & -\mathbf{I}^R & \mathbf{S}^R \\ -(\mathbf{S}^I)^T & \mathbf{D}^{II} & -(\mathbf{S}^{RI})^T & -(\mathbf{D}^{RI})^T \\ (\mathbf{I}^R)^T & \mathbf{S}^{RI} & -\mathbf{I}^{RR} & \mathbf{S}^{RR} \\ -(\mathbf{S}^R)^T & \mathbf{D}^{RI} & -(\mathbf{S}^{RR})^T & \mathbf{D}^{RR} \end{bmatrix}}{\text{Pf}\begin{bmatrix} 0 & \mathbf{S}^I \\ -(\mathbf{S}^I)^T & \mathbf{D}^{II} \end{bmatrix}}. \tag{2.87}$$

We again remind the reader that under the identification (2.64), the above 'finite-volume partition function' vanishes and the 'correlation function' diverges as its Pfaffian denominator vanishes.

III. SYMPLECTIC ENSEMBLE

For $\beta = 4$, we concentrate on even flavor cases for a technical reason, and treat the following two cases separately:

$$\mathbf{A} : \{m\} = (m_1, m_1, \ldots, m_\alpha, m_\alpha, -m_1, -m_1, \ldots, -m_\alpha, -m_\alpha),$$
$$\mathbf{B} : \{m\} = (m_1, m_1, \ldots, m_\alpha, m_\alpha, -m_1, -m_1, \ldots, -m_\alpha, -m_\alpha, 0, 0).$$

The $\beta = 4$ case with an odd number of fermions will not be treated in this Article.

A. $N_f = 0 \bmod 4$

We first consider the case with $N_f \equiv 4\alpha$ flavors and $\{m\}=(m_1,m_1,\ldots,m_\alpha,m_\alpha,-m_1,-m_1,\ldots,-m_\alpha,-m_\alpha)$. We express the partition function (1.12) of the RME in terms of eigenvalues $\{x_j\}$ of H:

$$Z(\{m\}) = \frac{1}{N!}\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \prod_{j=1}^N dx_j \prod_{j=1}^N \left(e^{-2x_j^2}\prod_{k=1}^\alpha (x_j^2 + m_k^2)\right)\prod_{j>k}^N (x_j - x_k)^4. \tag{3.1}$$

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The p-level correlation function of the matrix H is defined as

$$\rho(x_1, \ldots, x_p; \{m\}) = \Big\langle \prod_{j=1}^{p} \mathrm{tr}\, \delta(x_j - H) \Big\rangle$$

$$= \frac{\Xi_p(x_1, \ldots, x_p; \{m\})}{\Xi_0(\{m\})}, \tag{3.2}$$

$$\Xi_p(x_1, \ldots, x_p; \{m\}) = \frac{1}{(N-p)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=p+1}^{N} dx_j \prod_{j=1}^{N} \left(\mathrm{e}^{-2x_j^2} \prod_{k=1}^{\alpha} (x_j^2 + m_k^2) \right) \prod_{j>k}^{N} (x_j - x_k)^4 \tag{3.3}$$

$(\Xi_0 = Z)$. We define new variables z_j as

$$\left. \begin{array}{l} z_{2j-1} = im_j \\ z_{2j} = -im_j \end{array} \right\} \quad (j = 1, \ldots, \alpha). \tag{3.4}$$

Then the multiple integral (3.3) is expressed as

$$\Xi_p(x_1, \ldots, x_p; \{z\}) = \frac{1}{\prod_{j=1}^{2\alpha} w(z_j) \prod_{j>k}^{2\alpha} (z_j - z_k)}$$

$$\times \frac{1}{(N-p)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=2\alpha+p+1}^{2\alpha+N} dx_j \prod_{j=1}^{2\alpha} w(z_j) \prod_{j=1}^{N} w(x_j)^2 \prod_{j>k}^{2\alpha} (z_j - z_k) \prod_{j=1}^{N} \prod_{k=1}^{2\alpha} (x_j - z_k)^2 \prod_{j>k}^{N} (x_j - x_k)^4, \tag{3.5}$$

where $w(z) = \mathrm{e}^{-z^2}$.

Let us denote the integrand in Eq.(3.5) as

$$p(z_1, \ldots, z_{2\alpha}; x_1, \ldots, x_N) = \prod_{j=1}^{2\alpha} w(z_j) \prod_{j=1}^{N} w(x_j)^2 \prod_{j>k}^{2\alpha} (z_j - z_k) \prod_{j=1}^{N} \prod_{k=1}^{2\alpha} (x_j - z_k)^2 \prod_{j>k}^{N} (x_j - x_k)^4. \tag{3.6}$$

The above expressions can be represented as a quaternion determinant. In doing so, we introduce monic skew-orthogonal polynomials $Q_j(z) = z^j + \cdots$ that satisfy

$$\langle Q_{2j}, Q_{2k+1} \rangle_Q = -\langle Q_{2k+1}, Q_{2j} \rangle_Q = q_j \delta_{jk}, \tag{3.7}$$
$$\langle Q_{2j}, Q_{2k} \rangle_Q = \langle Q_{2j+1}, Q_{2k+1} \rangle_Q = 0,$$

where

$$\langle f, g \rangle_Q = \int_{-\infty}^{\infty} dz\, w(z)^2 \big(f(z) g'(z) - f'(z) g(z) \big). \tag{3.8}$$

Explicit forms for the skew-orthogonal polynomials and their norms associated with the Gaussian weight $w(z)$ are known in terms of the Hermite polynomials [40]:

$$Q_{2j}(z) = -\frac{1}{2^{3j+1/2}} \mathrm{e}^{z^2} \int_{-\infty}^{z} \mathrm{e}^{-z'^2} H_{2j+1}(\sqrt{2} z') dz',$$

$$Q_{2j+1}(z) = \frac{1}{2^{3j+3/2}} H_{2j+1}(\sqrt{2} z), \tag{3.9}$$

$$q_j = 2^{-4j-1/2} \sqrt{\pi} (2j+1)! \, .$$

It can be readily seen that (cf. Eq.(26) of Ref. [29]) $p(z_1, \ldots, z_{2\alpha}; x_1, \ldots, x_N)$ is represented as a Pfaffian:

$$p(z_1, \ldots, z_{2\alpha}; x_1, \ldots, x_N) = \det \begin{bmatrix} \Psi_0(z_1) & \Psi_1(z_1) & \cdots & \Psi_{2N+2\alpha-1}(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_0(z_{2\alpha}) & \Psi_1(z_{2\alpha}) & \cdots & \Psi_{2N+2\alpha-1}(z_{2\alpha}) \\ \Psi_0'(x_1) & \Psi_1'(x_1) & \cdots & \Psi_{2N+2\alpha-1}'(x_1) \\ \Psi_0(x_1) & \Psi_1(x_1) & \cdots & \Psi_{2N+2\alpha-1}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_0'(x_N) & \Psi_1'(x_N) & \cdots & \Psi_{2N+2\alpha-1}'(x_N) \\ \Psi_0(x_N) & \Psi_1(x_N) & \cdots & \Psi_{2N+2\alpha-1}(x_N) \end{bmatrix}$$

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$$= \Big(\prod_{j=0}^{\alpha+N-1} q_j \Big) \mathrm{Pf} \left[\begin{array}{cc} \left[\begin{array}{cc} -I(z_{2j}, z_{2k}) & I(z_{2j}, z_{2k-1}) \\ I(z_{2j-1}, z_{2k}) & -I(z_{2j-1}, z_{2k-1}) \end{array} \right]_{j,k=1,\ldots,\alpha} & \left[\begin{array}{cc} S(z_{2j}, x_k) & I(z_{2j}, x_k) \\ -S(z_{2j-1}, x_k) & -I(z_{2j-1}, x_k) \end{array} \right]_{j=1,\ldots,\alpha;k=1,\ldots,N} \\ \left[\begin{array}{cc} -S(z_{2k}, x_j) & S(z_{2k-1}, x_j) \\ I(x_j, z_{2k}) & -I(x_j, z_{2k-1}) \end{array} \right]_{j=1,\ldots,N;k=1,\ldots,\alpha} & \left[\begin{array}{cc} D(x_j, x_k) & S(x_k, x_j) \\ -S(x_j, x_k) & -I(x_j, x_k) \end{array} \right]_{j,k=1,\ldots,N} \end{array} \right]. \tag{3.10}$$

Here

$$\Psi_j(z) = w(z) Q_j(z), \tag{3.11}$$

and the functions $S(x, y)$, $D(x, y)$, $I(x, y)$ are given by

$$S(x, y) = \sum_{j=0}^{\alpha+N-1} \frac{\Psi_{2j}(x) \Psi'_{2j+1}(y) - \Psi_{2j+1}(x) \Psi'_{2j}(y)}{q_j},$$

$$D(x, y) = \sum_{j=0}^{\alpha+N-1} \frac{\Psi'_{2j}(x) \Psi'_{2j+1}(y) - \Psi'_{2j+1}(x) \Psi'_{2j}(y)}{q_j} = \frac{\partial}{\partial x} S(x, y), \tag{3.12}$$

$$I(x, y) = - \sum_{j=0}^{\alpha+N-1} \frac{\Psi_{2j}(x) \Psi_{2j+1}(y) - \Psi_{2j+1}(x) \Psi_{2j}(y)}{q_j} = - \int_x^y S(x, z) dz.$$

Using Dyson's identity (A.6), we can convert the Pfaffian into a quaternion determinant:

$$p(z_1, \ldots, z_{2\alpha}; x_1, \ldots, x_N) = \Big(\prod_{j=0}^{\alpha+N-1} q_j \Big) \mathrm{Tdet} \left[\begin{array}{cc} [g(z_{2j-1}, z_{2j}; z_{2k-1}, z_{2k})]_{j,k=1,\ldots,\alpha} & [h(z_{2j-1}, z_{2j}; x_k)]_{j=1,\ldots,\alpha;k=1,\ldots,N} \\ [\hat{h}(z_{2k-1}, z_{2k}; x_j)]_{j=1,\ldots,N;k=1,\ldots,\alpha} & [f(x_j, x_k)]_{j,k=1,\ldots,N} \end{array} \right], \tag{3.13}$$

where

$$g(z_{2j-1}, z_{2j}; z_{2k-1}, z_{2k}) = \left[\begin{array}{cc} -I(z_{2j-1}, z_{2k}) & I(z_{2j-1}, z_{2k-1}) \\ -I(z_{2j}, z_{2k}) & I(z_{2j}, z_{2k-1}) \end{array} \right],$$

$$h(z_{2j-1}, z_{2j}; x_k) = \left[\begin{array}{cc} S(z_{2j-1}, x_k) & I(z_{2j-1}, x_k) \\ S(z_{2j}, x_k) & I(z_{2j}, x_k) \end{array} \right], \tag{3.14}$$

$$f(x_j, x_k) = \left[\begin{array}{cc} S(x_j, x_k) & I(x_j, x_k) \\ D(x_j, x_k) & S(x_k, x_j) \end{array} \right],$$

and $\hat{h}$ stands for the dual of h (see Appendix A). The skew-orthogonality relations (3.7) lead to

$$\int_{-\infty}^{\infty} f(x', x) f(x, x'') dx = f(x', x''),$$

$$\int_{-\infty}^{\infty} h(z, z'; x) f(x, x') dx = h(z, z'; x'), \tag{3.15}$$

$$\int_{-\infty}^{\infty} h(z, z', x) \hat{h}(w, w'; x) dx = g(z, z'; w, w'),$$

which mean that the condition of *Theorem 3* is satisfied. Therefore we can inductively write

$$\Xi_p(x_1, \ldots, x_p; z_1, \ldots, z_{2\alpha}) =$$
$$\frac{\prod_{j=0}^{\alpha+N-1} q_j}{\prod_{j=1}^{2\alpha} w(z_j) \prod_{j>k}^{2\alpha} (z_j - z_k)} \mathrm{Tdet} \left[\begin{array}{cc} [g(z_{2j-1}, z_{2j}; z_{2k-1}, z_{2k})]_{j,k=1,\ldots,\alpha} & [h(z_{2j-1}, z_{2j}; x_k)]_{j=1,\ldots,\alpha;k=1,\ldots,p} \\ [\hat{h}(z_{2k-1}, z_{2k}; x_j)]_{j=1,\ldots,p;k=1,\ldots,\alpha} & [f(x_j, x_k)]_{j,k=1,\ldots,p} \end{array} \right]. \tag{3.16}$$

Then the p-level correlation function (3.2) is written as

$$\rho(x_1,\ldots,x_p;\{m\}) = \frac{\Xi_p(x_1,\ldots,x_p;z_1,\ldots,z_{2n})}{\Xi_0(z_1,\ldots,z_{2n})}$$

$$= \frac{\mathrm{Tdet}\left[\begin{array}{cc} [g(z_{2j-1},z_{2j};z_{2k-1},z_{2k})]_{j,k=1,\ldots,\alpha} & [h(z_{2j-1},z_{2j};x_k)]_{j=1,\ldots,\alpha;k=1,\ldots,p} \\ [\hat{h}(z_{2k-1},z_{2k};x_j)]_{j=1,\ldots,p;k=1,\ldots,\alpha} & [f(x_j,x_k)]_{j,k=1,\ldots,p} \end{array}\right]}{\mathrm{Tdet}[g(z_{2j-1},z_{2j};z_{2k-1},z_{2k})]_{j,k=1,\ldots,\alpha}}. \tag{3.17}$$

We introduce a set of notations

$$\begin{aligned} S_{jk}^{II} &= S(z_j,z_k) & (j,k=1,\ldots,2\alpha), \\ S_{jk}^{IR} &= S(z_j,x_k) & (j=1,\ldots,2\alpha;\ k=1,\ldots,p), \\ S_{jk}^{RR} &= S(x_j,x_k) & (j,k=1,\ldots,p), \end{aligned} \tag{3.18}$$

and similarly for D and I. Using Dyson's identity (A.6) again, we can convert the quaternion determinant back to a Pfaffian, so that the correlation functions read

$$\rho(x_1,\ldots,x_p;\{m\}) = (-1)^{p(p-1)/2} \frac{\mathrm{Pf}\left[\begin{array}{ccc} -I^{II} & -I^{IR} & S^{IR} \\ (I^{IR})^T & -I^{RR} & S^{RR} \\ -(S^{IR})^T & -(S^{RR})^T & D^{RR} \end{array}\right]}{\mathrm{Pf}[-I^{II}]}. \tag{3.19}$$

Now we proceed to evaluate the component functions of the quaternion kernel, which are expressed as (derivatives or integrals of) the analytic function $S(x,y)$ with real and/or imaginary arguments, in the asymptotic limit where unfolded microscopic variables

$$\begin{aligned} \sqrt{2N}z_{2j-1} \equiv \zeta_{2j-1} &= i\mu_j & (j=1,\ldots,\alpha), \\ \sqrt{2N}z_{2j} \equiv \zeta_{2j} &= -i\mu_j & (j=1,\ldots,\alpha), \\ \sqrt{2N}x_j \equiv \lambda_j & & (j=1,\ldots,p), \end{aligned} \tag{3.20}$$

are kept fixed. We note that $S(x,y)$ has a compact expression

$$\begin{aligned} S(x,y) &= \frac{e^{-x^2-y^2}}{2^{2\alpha+2N+1}\sqrt{\pi}\Gamma(2\alpha+2N)} \frac{H_{2\alpha+2N}(\sqrt{2}x)H_{2\alpha+2N-1}(\sqrt{2}y) - H_{2\alpha+2N-1}(\sqrt{2}x)H_{2\alpha+2N}(\sqrt{2}y)}{x-y} \\ &\quad + \frac{e^{-y^2}}{2^{2\alpha+2N}\sqrt{\pi}\Gamma(2\alpha+2N)} H_{2\alpha+2N}(\sqrt{2}y) \int_{-\infty}^{x} e^{-u^2} H_{2\alpha+2N-1}(\sqrt{2}u) du. \end{aligned} \tag{3.21}$$

In the second line of Eq.(3.21), we have singled out the unitary scalar kernel and applied to it the Christoffel-Darboux formula. Substituting asymptotic formulas for the Hermite polynomials (2.38), one can show that $S(x,y)$ approaches the sine kernel [40]:

$$\frac{1}{\sqrt{2N}}S\left(\frac{\zeta}{\sqrt{2N}},\frac{\zeta'}{\sqrt{2N}}\right) \sim \frac{\sin 2(\zeta-\zeta')}{2\pi(\zeta-\zeta')} = K(2(\zeta-\zeta')). \tag{3.22}$$

The Pfaffian elements in the the asymptotic limit, after taking into account an unfolding by the factor $\sqrt{2N}$, are then expressed in terms of $K(\zeta)$:

$$\mathbf{I}_{jk}^{II} \equiv I(z_j,z_k) \sim \int_0^{\zeta_j-\zeta_k} K(2\zeta)d\zeta, \tag{3.23a}$$

$$\mathbf{S}_{jk}^{IR} \equiv \frac{1}{\sqrt{2N}}S(z_j,x_k) \sim K(2(\zeta_j-\lambda_k)), \tag{3.23b}$$

$$\mathbf{I}_{jk}^{IR} \equiv I(z_j,x_k) \sim \int_0^{\zeta_j-\lambda_k} K(2\zeta)d\zeta, \tag{3.23c}$$

$$\mathbf{S}_{jk}^{RR} \equiv \frac{1}{\sqrt{2N}}S(x_j,x_k) \sim K(2(\lambda_j-\lambda_k)), \tag{3.23d}$$

$$\mathbf{D}_{jk}^{RR} \equiv \frac{1}{2N}D(x_j,x_k) \sim 2K'(2(\lambda_j-\lambda_k)), \tag{3.23e}$$

$$\mathbf{I}_{jk}^{RR} \equiv I(x_j,x_k) \sim \int_0^{\lambda_j-\lambda_k} K(2\lambda)d\lambda. \tag{3.23f}$$

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These matrix elements constitute the finite-volume partition function

$$\mathcal{Z}(\{m\}) \equiv \Xi_0(\{\frac{\mu}{\sqrt{2N}}\})$$

$$= \text{const.} \frac{\text{Pf}[-\mathbf{I}^{II}]}{\prod_{j=1}^{\alpha} \mu_j \prod_{j>k}^{\alpha} (\mu_j^2 - \mu_k^2)^2}, \tag{3.24}$$

and the scaled spectral correlation functions

$$\rho_s(\lambda_1, \ldots, \lambda_p; \{\mu\}) \equiv \left(\frac{1}{\sqrt{2N}}\right)^p \rho(\frac{\lambda_1}{\sqrt{2N}}, \ldots, \frac{\lambda_p}{\sqrt{2N}}; \{\frac{\mu}{\sqrt{2N}}\})$$

$$= (-1)^{p(p-1)/2} \frac{\text{Pf} \begin{bmatrix} -\mathbf{I}^{II} & -\mathbf{I}^{IR} & \mathbf{S}^{IR} \\ (\mathbf{I}^{IR})^T & -\mathbf{I}^{RR} & \mathbf{S}^{RR} \\ -(\mathbf{S}^{IR})^T & -(\mathbf{S}^{RR})^T & \mathbf{D}^{RR} \end{bmatrix}}{\text{Pf}[-\mathbf{I}^{II}]}. \tag{3.25}$$

In the quenched limit $\mu_1, \ldots, \mu_\alpha \to \infty$ when the ratio of two Pfaffians is replaced by a minor $\text{Pf} \begin{bmatrix} -\mathbf{I}^{RR} & \mathbf{S}^{RR} \\ -(\mathbf{S}^{RR})^T & \mathbf{D}^{RR} \end{bmatrix}$, the correlation functions approach those of the Gaussian symplectic ensemble [40]. By the same token, it satisfies a sequence

$$\rho_s(\{\lambda\}; \mu_1, \mu_1, \ldots, \mu_n, \mu_n, -\mu_1, -\mu_1, \ldots, -\mu_n, -\mu_n) \xrightarrow{\mu_n \to \infty} \tag{3.26}$$

$$\rho_s(\{\lambda\}; \mu_1, \mu_1, \ldots, \mu_{n-1}, \mu_{n-1}, -\mu_1, -\mu_1, \ldots, -\mu_{n-1}, -\mu_{n-1}) \xrightarrow{\mu_{n-1} \to \infty} \cdots,$$

as each of the masses is decoupled by going to infinity. To illustrate this decoupling, we exhibit in FIG.3 a plot of the spectral density $\rho_s(\lambda; \mu, \mu, -\mu, -\mu)$ $(p = 1, \alpha = 1)$.

B. $N_f = 2 \mod 4$

Next we consider the case with $N_f \equiv 4\alpha + 2$ flavors and $\{m\} = (m_1, m_1, \ldots, m_\alpha, m_\alpha, m_1, -m_1, \ldots, -m_\alpha, -m_\alpha, 0, 0)$. We express the partition function (1.12) of the RME in terms of eigenvalues $\{x_j\}$ of H

$$Z(\{m\}) = \frac{1}{N!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} dx_j \prod_{j=1}^{N} \left(e^{-2x_j^2} x_j^2 \prod_{k=1}^{\alpha} (x_j^2 + m_k^2)\right) \prod_{j>k}^{N} (x_j - x_k)^4. \tag{3.27}$$

The p-level correlation function of the matrix H is defined as

$$\rho(x_1, \ldots, x_p; \{m\}) = \langle \prod_{j=1}^{p} \text{tr}\,\delta(x_j - H) \rangle$$

$$= \frac{\Xi_p(x_1, \ldots, x_p; \{m\})}{\Xi_0(\{m\})}, \tag{3.28}$$

$$\Xi_p(x_1, \ldots, x_p; \{m\}) = \frac{1}{(N-p)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=p+1}^{N} dx_j \prod_{j=1}^{N} \left(e^{-2x_j^2} x_j^2 \prod_{k=1}^{\alpha} (x_j^2 + m_k^2)\right) \prod_{j>k}^{N} (x_j - x_k)^4 \tag{3.29}$$

$(\Xi_0 = Z)$. We define new variables z_j as

$$z_0 = 0,$$
$$\left.\begin{array}{l} z_{2j-1} = im_j \\ z_{2j} = -im_j \end{array}\right\} \quad (j = 1, \ldots, \alpha). \tag{3.30}$$

Then the multiple integral (3.29) is expressed as

$$\Xi_p(x_1, \ldots, x_p; \{z\}) = \frac{1}{\prod_{j=0}^{2\alpha} w(z_j) \prod_{j>k\geq 0}^{2\alpha} (z_j - z_k)}$$

$$\times \frac{1}{(N-p)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=2\alpha+p+1}^{2\alpha+N} dx_j \prod_{j=0}^{2\alpha} w(z_j) \prod_{j=1}^{N} w(x_j)^2 \prod_{j>k\geq 0}^{2\alpha} (z_j - z_k) \prod_{j=1}^{N} \prod_{k=0}^{2\alpha} (x_j - z_k)^2 \prod_{j>k}^{N} (x_j - x_k)^4, \tag{3.31}$$

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where $w(z) = e^{-z^2}$.

Let us denote the integrand in Eq.(3.31) as

$$p(z_0, \ldots, z_{2\alpha}; x_1, \ldots, x_N) = \prod_{j=0}^{2\alpha} w(z_j) \prod_{j=1}^{N} w(x_j)^2 \prod_{j>k\geq 0}^{2\alpha} (z_j - z_k) \prod_{j=1}^{N} \prod_{k=0}^{2\alpha} (x_j - z_k)^2 \prod_{j>k}^{N} (x_j - x_k)^4. \tag{3.32}$$

It can be readily seen that $p(z_0, \ldots, z_{2\alpha}; x_1, \ldots, x_N)$ is represented as a Pfaffian:

$$p(z_0, \ldots, z_{2\alpha}; x_1, \ldots, x_N) = \det
\begin{vmatrix}
\Psi_0(z_0) & \Psi_1(z_0) & \cdots & \Psi_{2N+2\alpha}(z_0) \\
\Psi_0(z_1) & \Psi_1(z_1) & \cdots & \Psi_{2N+2\alpha}(z_1) \\
\vdots & \vdots & \ddots & \vdots \\
\Psi_0(z_{2\alpha}) & \Psi_1(z_{2\alpha}) & \cdots & \Psi_{2N+2\alpha}(z_{2\alpha}) \\
\Psi_0'(x_1) & \Psi_1'(x_1) & \cdots & \Psi_{2N+2\alpha}'(x_1) \\
\Psi_0(x_1) & \Psi_1(x_1) & \cdots & \Psi_{2N+2\alpha}(x_1) \\
\vdots & \vdots & \ddots & \vdots \\
\Psi_0'(x_N) & \Psi_1'(x_N) & \cdots & \Psi_{2N+2\alpha}'(x_N) \\
\Psi_0(x_N) & \Psi_1(x_N) & \cdots & \Psi_{2N+2\alpha}(x_N)
\end{vmatrix}$$

$$= \left(\prod_{j=0}^{\alpha+N-1} q_j \right) \mathrm{Pf}
\begin{bmatrix}
[G_{jk}]_{j,k=1,\ldots,\alpha} & [A_j]_{j=1,\ldots,\alpha} & [H_{jk}]_{j=1,\ldots,\alpha; k=1,\ldots,N} \\
[\hat{A}_k]_{k=1,\ldots,\alpha} & \Omega & [B_k]_{k=1,\ldots,N} \\
[\hat{H}_{kj}]_{j=1,\ldots,N; k=1,\ldots,\alpha} & [\hat{B}_j]_{j=1,\ldots,N} & [F_{jk}]_{j,k=1,\ldots,N}
\end{bmatrix}, \tag{3.33}$$

where

$$G_{jk} = \begin{bmatrix} -\tilde{I}(z_{2j}, z_{2k}) & \tilde{I}(z_{2j}, z_{2k-1}) \\ \tilde{I}(z_{2j-1}, z_{2k}) & -\tilde{I}(z_{2j-1}, z_{2k-1}) \end{bmatrix},$$

$$H_{jk} = \begin{bmatrix} \tilde{S}(z_{2j}, x_k) & \tilde{I}(z_{2j}, x_k) \\ -\tilde{S}(z_{2j-1}, x_k) & -\tilde{I}(z_{2j-1}, x_k) \end{bmatrix},$$

$$F_{jk} = \begin{bmatrix} \tilde{D}(x_j, x_k) & \tilde{S}(x_k, x_j) \\ -\tilde{S}(x_j, x_k) & -\tilde{I}(x_j, x_k) \end{bmatrix},$$

$$A_j = \begin{bmatrix} \Psi_{2N+2\alpha}(z_{2j}) & \tilde{I}(z_{2j}, z_0) \\ -\Psi_{2N+2\alpha}(z_{2j-1}) & -\tilde{I}(z_{2j-1}, z_0) \end{bmatrix}, \tag{3.34}$$

$$B_k = \begin{bmatrix} -\Psi_{2N+2\alpha}'(x_k) & \Psi_{2N+2\alpha}(x_k) \\ -\tilde{S}(z_0, x_k) & -\tilde{I}(z_0, x_k) \end{bmatrix},$$

$$\Omega = \begin{bmatrix} 0 & \Psi_{2N+2\alpha}(z_0) \\ -\Psi_{2N+2\alpha}(z_0) & 0 \end{bmatrix}.$$

The functions $\tilde{S}$, $\tilde{D}$, $\tilde{I}$ are given in terms of S, D and I defined in Eq.(3.12) according to

$$\tilde{S}(x, y) = S(x, y)\Big|_* + \frac{\Psi_{2\alpha+2N}(x)}{s_{2\alpha+2N}},$$

$$\tilde{D}(x, y) = D(x, y)\Big|_*, \tag{3.35}$$

$$\tilde{I}(x, y) = I(x, y)\Big|_* + \frac{\Psi_{2\alpha+2N}'(x) - \Psi_{2\alpha+2N}'(y)}{s_{2\alpha+2N}},$$

and

$$s_j = \int_{-\infty}^{\infty} \Psi_j(x)\,dx. \tag{3.36}$$

Here $*$ stands for a set of substitutions

$$Q_j(z) \mapsto Q_j(z) - \frac{s_j}{s_{2\alpha+2N}} Q_{2\alpha+2N}(z) \qquad (j = 0, \ldots, 2\alpha + 2N - 1). \tag{3.37}$$

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Using Dyson's identity (A.6), we find

$$p(z_0, \ldots, z_{2\alpha}; x_1, \ldots, x_N)$$
$$= \left(\prod_{j=0}^{\alpha+N-1} q_j \right) \mathrm{Tdet} \left[\begin{array}{ccc} [g(z_{2j-1}, z_{2j}; z_{2k-1}, z_{2k})]_{j,k=1,\ldots,\alpha} & [a(z_{2j-1}, z_{2j})]_{j=1,\ldots,\alpha} & [h(z_{2j-1}, z_{2j}; x_k)]_{j=1,\ldots,\alpha; k=1,\ldots,N} \\ [\hat{a}(z_{2k-1}, z_{2k})]_{k=1,\ldots,\alpha} & \omega & [b(x_k)]_{k=1,\ldots,N} \\ [\hat{h}(z_{2k-1}, z_{2k}; x_j)]_{j=1,\ldots,N; k=1,\ldots,\alpha} & [\hat{b}(x_j)]_{j=1,\ldots,N} & [f(x_j, x_k)]_{j,k=1,\ldots,N} \end{array} \right],$$

(3.38)

where

$$g(z_{2j-1}, z_{2j}; z_{2k-1}, z_{2k}) = \left[\begin{array}{cc} -\tilde{I}(z_{2j-1}, z_{2k}) & \tilde{I}(z_{2j-1}, z_{2k-1}) \\ -\tilde{I}(z_{2j}, z_{2k}) & \tilde{I}(z_{2j}, z_{2k-1}) \end{array} \right],$$

$$h(z_{2j-1}, z_{2j}; x_k) = \left[\begin{array}{cc} \tilde{S}(z_{2j-1}, x_k) & \tilde{I}(z_{2j-1}, x_k) \\ \tilde{S}(z_{2j}, x_k) & \tilde{I}(z_{2j}, x_k) \end{array} \right],$$

$$f(x_j, x_k) = \left[\begin{array}{cc} \tilde{S}(x_j, x_k) & \tilde{I}(x_j, x_k) \\ \tilde{D}(x_j, x_k) & \tilde{S}(x_k, x_j) \end{array} \right],$$

$$a(z_{2j-1}, z_{2j}) = \left[\begin{array}{cc} \Psi_{2N+2\alpha}(z_{2j-1}) & \tilde{I}(z_{2j-1}, z_0) \\ \Psi_{2N+2\alpha}(z_{2j}) & \tilde{I}(z_{2j}, z_0) \end{array} \right],$$

(3.39)

$$b(x_k) = \left[\begin{array}{cc} \tilde{S}(z_0, x_k) & \tilde{I}(z_0, x_k) \\ -\Psi'_{2N+2\alpha}(x_k) & \Psi_{2N+2\alpha}(x_k) \end{array} \right],$$

$$\omega = \left[\begin{array}{cc} \Psi_{2N+2\alpha}(z_0) & 0 \\ 0 & \Psi_{2N+2\alpha}(z_0) \end{array} \right],$$

and $\hat{a}$, $\hat{b}$, $\hat{h}$ stand for the duals of a, b, h. The skew-orthogonality relations (3.7) lead to

$$\int_{-\infty}^{\infty} f(x', x) f(x, x'') dx = f(x', x''),$$

$$\int_{-\infty}^{\infty} b(x) f(x, x') dx = b(x'),$$

$$\int_{-\infty}^{\infty} h(z, z'; x) f(x, x') dx = h(z, z'; x'),$$

$$\int_{-\infty}^{\infty} b(x) \hat{b}(x) dx = \omega,$$

(3.40)

$$\int_{-\infty}^{\infty} h(z, z'; x) \hat{b}(x) dx = a(z, z'),$$

$$\int_{-\infty}^{\infty} h(z, z', x) \hat{h}(w, w'; x) dx = g(z, z'; w, w'),$$

which mean that the condition of *Theorem 3* is satisfied. Therefore we can inductively write

$$\Xi_p(x_1, \ldots, x_p; z_0, \ldots, z_{2\alpha}) = \frac{\prod_{j=0}^{\alpha+N-1} q_j}{\prod_{j=0}^{2\alpha} w(z_j) \prod_{j>k\geq 0}^{2\alpha} (z_j - z_k)}$$
$$\times \mathrm{Tdet} \left[\begin{array}{ccc} [g(z_{2j-1}, z_{2j}; z_{2k-1}, z_{2k})]_{j,k=1,\ldots,\alpha} & [a(z_{2j-1}, z_{2j})]_{j=1,\ldots,\alpha} & [h(z_{2j-1}, z_{2j}; x_k)]_{j=1,\ldots,\alpha; k=1,\ldots,p} \\ [\hat{a}(z_{2k-1}, z_{2k})]_{k=1,\ldots,\alpha} & \omega & [b(x_k)]_{k=1,\ldots,p} \\ [\hat{h}(z_{2k-1}, z_{2k}; x_j)]_{j=1,\ldots,p; k=1,\ldots,\alpha} & [\hat{b}(x_j)]_{j=1,\ldots,p} & [f(x_j, x_k)]_{j,k=1,\ldots,p} \end{array} \right].$$

(3.41)

Then the p-level correlation function (3.28) is written as

$$\rho(x_1, \ldots, x_p; \{m\}) = \frac{\Xi_p(x_1, \ldots, x_p; z_0, \ldots, z_{2n})}{\Xi_0(z_0, \ldots, z_{2n})}$$

$$= \frac{\mathrm{Tdet} \left[\begin{array}{ccc} [g(z_{2j-1}, z_{2j}; z_{2k-1}, z_{2k})]_{j,k=1,\ldots,\alpha} & [a(z_{2j-1}, z_{2j})]_{j=1,\ldots,\alpha} & [h(z_{2j-1}, z_{2j}; x_k)]_{j=1,\ldots,\alpha; k=1,\ldots,p} \\ [\hat{a}(z_{2k-1}, z_{2k})]_{k=1,\ldots,\alpha} & \omega & [b(x_k)]_{k=1,\ldots,p} \\ [\hat{h}(z_{2k-1}, z_{2k}; x_j)]_{j=1,\ldots,p; k=1,\ldots,\alpha} & [\hat{b}(x_j)]_{j=1,\ldots,p} & [f(x_j, x_k)]_{j,k=1,\ldots,p} \end{array} \right]}{\mathrm{Tdet} \left[\begin{array}{cc} [g(z_{2j-1}, z_{2j}; z_{2k-1}, z_{2k})]_{j,k=1,\ldots,\alpha} & [a(z_{2j-1}, z_{2j})]_{j=1,\ldots,\alpha} \\ [\hat{a}(z_{2k-1}, z_{2k})]_{k=1,\ldots,\alpha} & \omega \end{array} \right]}.$$

(3.42)

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Now we make replacement of the elements of the quaternion kernel back to those defined in Eq.(3.12):

$$\tilde{S}(x,y) \to S(x,y), \quad \tilde{D}(x,y) \to D(x,y), \quad \tilde{I}(x,y) \to I(x,y), \tag{3.43}$$

which does not change the values of the quaternion determinants in Eqs.(3.41) and (3.42). We introduce a set of notations

$$
\begin{aligned}
S^{II}_{jk} &= S(z_j, z_k) && (j,k = 0, \ldots, 2\alpha), \\
S^{IR}_{jk} &= S(z_j, x_k) && (j = 0, \ldots, 2\alpha; \; k = 1, \ldots, p), \\
S^{RR}_{jk} &= S(x_j, x_k) && (j,k = 1, \ldots, p),
\end{aligned}
\tag{3.44}
$$

and similarly for D and I, and

$$Q^I_j = \frac{\Psi_{2N+2\alpha}(z_j)}{s_{2N+2\alpha}} \quad (j = 0, \ldots, 2\alpha),$$

$$Q^R_j = \frac{\Psi_{2N+2\alpha}(x_j)}{s_{2N+2\alpha}} \quad (j = 1, \ldots, p), \tag{3.45}$$

$$P^R_j = \frac{\Psi'_{2N+2\alpha}(x_j)}{s_{2N+2\alpha}} \quad (j = 1, \ldots, p). \tag{3.46}$$

Using Dyson's identity (A.6), we can convert the quaternion determinant back to a Pfaffian, so that the correlation functions read

$$\rho(x_1, \ldots, x_p; \{m\}) = (-1)^{p(p-1)/2} \frac{\mathrm{Pf}\begin{bmatrix} -I^{II} & Q^I & -I^{IR} & S^{IR} \\ -(Q^I)^T & 0 & -(Q^R)^T & -(P^R)^T \\ (I^{IR})^T & Q^R & -I^{RR} & S^{RR} \\ -(S^{IR})^T & P^R & -(S^{RR})^T & D^{RR} \end{bmatrix}}{\mathrm{Pf}\begin{bmatrix} -I^{II} & Q^I \\ -(Q^I)^T & 0 \end{bmatrix}}. \tag{3.47}$$

Now we proceed to evaluate the component functions in the asymptitic limit where unfolded microscopic variables

$$
\begin{aligned}
\sqrt{2N}z_0 &\equiv \zeta_0 = 0, \\
\sqrt{2N}z_{2j-1} &\equiv \zeta_{2j-1} = i\mu_j && (j = 1, \ldots, \alpha), \\
\sqrt{2N}z_{2j} &\equiv \zeta_{2j} = -i\mu_j && (j = 1, \ldots, \alpha), \\
\sqrt{2N}x_j &\equiv \lambda_j && (j = 1, \ldots, p),
\end{aligned}
\tag{3.48}
$$

are kept fixed. We obtain Eq.(3.23) and

$$\mathbf{Q}^I_j \equiv \frac{1}{\sqrt{2N}} Q^I_j \sim \frac{1}{2}, \tag{3.49a}$$

$$\mathbf{Q}^R_j \equiv \frac{1}{\sqrt{2N}} Q^R_j \sim \frac{1}{2}, \tag{3.49b}$$

$$\mathbf{P}^R_j \equiv P^R_j \sim 0. \tag{3.49c}$$

These matrix elements constitute the finite-volume partition function

$$\mathcal{Z}(\{\mu\}) \equiv \Xi_0(\{\frac{\mu}{\sqrt{2N}}\})$$

$$= \mathrm{const.} \frac{\mathrm{Pf}\begin{bmatrix} -\mathbf{I}^{II} & \mathbf{Q}^I \\ -(\mathbf{Q}^I)^T & 0 \end{bmatrix}}{\prod_{j=1}^{\alpha} \mu_j^3 \prod_{j>k}^{\alpha} (\mu_j^2 - \mu_k^2)^2}, \tag{3.50}$$

and the scaled spectral correlation functions

$$\rho_s(\lambda_1,\ldots,\lambda_p;\{\mu\}) \equiv \left(\frac{1}{\sqrt{2N}}\right)^p \rho(\frac{\lambda_1}{\sqrt{2N}},\ldots,\frac{\lambda_p}{\sqrt{2N}};\{\frac{\mu}{\sqrt{2N}}\})$$

$$= (-1)^{p(p-1)/2} \frac{\mathrm{Pf}\begin{bmatrix} -\mathbf{I}^{II} & \mathbf{Q}^I & -\mathbf{I}^{IR} & \mathbf{S}^{IR} \\ -(\mathbf{Q}^I)^T & 0 & -(\mathbf{Q}^R)^T & -(\mathbf{P}^R)^T \\ (\mathbf{I}^{IR})^T & \mathbf{Q}^R & -\mathbf{I}^{RR} & \mathbf{S}^{RR} \\ -(\mathbf{S}^{IR})^T & \mathbf{P}^R & -(\mathbf{S}^{RR})^T & \mathbf{D}^{RR} \end{bmatrix}}{\mathrm{Pf}\begin{bmatrix} -\mathbf{I}^{II} & \mathbf{Q}^I \\ -(\mathbf{Q}^I)^T & 0 \end{bmatrix}}. \tag{3.51}$$

It satisfies a sequence

$$\rho_s(\{\lambda\};\mu_1,\mu_1,\ldots,\mu_n,\mu_n,-\mu_1,-\mu_1,\ldots,-\mu_n,-\mu_n,0,0) \overset{\mu_n\to\infty}{\longrightarrow} \tag{3.52}$$

$$\rho_s(\{\lambda\};\mu_1,\mu_1,\ldots,\mu_{n-1},\mu_{n-1},-\mu_1,-\mu_1,\ldots,-\mu_{n-1},-\mu_{n-1},0,0) \overset{\mu_{n-1}\to\infty}{\longrightarrow} \cdots,$$

as each of the masses is decoupled by going to infinity. To illustrate this decoupling, we exhibit in FIG.4 a plot of the spectral density $\rho_s(\lambda;\mu,\mu,-\mu,-\mu,0,0)$ ($p=1, \alpha=1$).

APPENDIX A: QUATERNION DETERMINANT

A quaternion is defined as a linear combination of four basic units $\{1, e_1, e_2, e_3\}$:

$$q = q_0 + \mathbf{q}\cdot\mathbf{e} = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3. \tag{A.1}$$

Here the coefficients q_0, q_1, q_2, and q_3 are real or complex numbers. The first part q_0 is called the scalar part of q. The quaternion basic units satisfy the multiplication laws

$$1\cdot 1 = 1, \quad 1\cdot e_j = e_j\cdot 1 = e_j \quad (j=1,2,3),$$
$$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1. \tag{A.2}$$

The multiplication is associative and in general not commutative. The dual $\hat{q}$ of a quaternion q is defined as

$$\hat{q} = q_0 - \mathbf{q}\cdot\mathbf{e}. \tag{A.3}$$

For a selfdual $N \times N$ matrix Q with quaternion elements q_{jk} has a dual matrix $\hat{Q} = [\hat{q}_{kj}]$. The quaternion units can be represented as 2×2 matrices

$$1 \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$e_2 \to \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad e_3 \to \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{A.4}$$

We define a quaternion determinant Tdet of a selfdual Q (i.e., $Q = \hat{Q}$) as

$$\mathrm{Tdet}\,Q = \sum_{P\in S_N} (-1)^{N-\ell} \prod_1^\ell (q_{ab}q_{bc}\cdots q_{da})_0, \tag{A.5}$$

where P denotes any permutation of the indices $(1,\ldots,N)$ consisting of ℓ exclusive cycles of the form $(a \to b \to c \to \cdots \to d \to a)$ and $(-1)^{N-\ell}$ is the parity of P. The subscript 0 means that the scalar part of the product is taken over each cycle. Note that a quaternion determinant of a selfdual quaternion matrix is always a scalar. The quaternion determinant of Q can as well be represented by its $2N \times 2N$ complex matrix representation $C(Q)$ [39]:

$$\mathrm{Tdet}Q = \mathrm{Pf}[JC(Q)], \quad J = \mathbb{1}_N \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{A.6}$$

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†   We adopt Cartan's convention on Riemannian symmetric spaces [19] and supplement subscripts to indicate the ranks of Lie group constituents.

‡   We tacitly assume that a regularization employed in convoluting the operators ε and K in Eq.(1.21) does not involve non-universality. For a detailed discussion, see Ref. [35].

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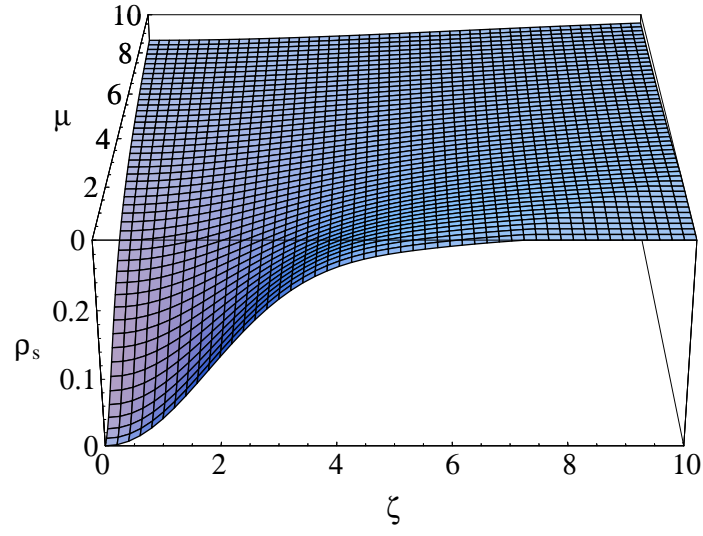[44] P.J. Forrester, T. Nagao, and G. Honner, Nucl. Phys. **B533**, 601 (1999).

FIG. 1. The scaled spectral density $\rho_s(\zeta; \mu, -\mu)$ for the orthogonal ensemble with two flavors ($n = 1$).
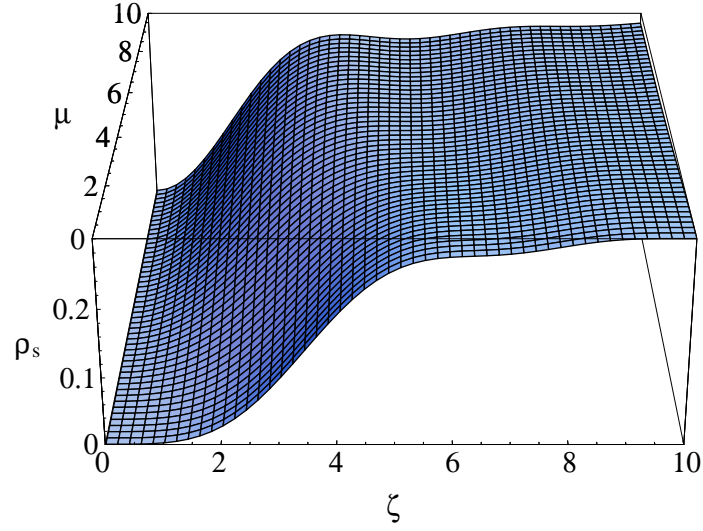

FIG. 2. The scaled spectral density $\rho_s(\zeta; \mu, -\mu, 0)$ for the orthogonal ensemble with three flavors ($n = 1$).
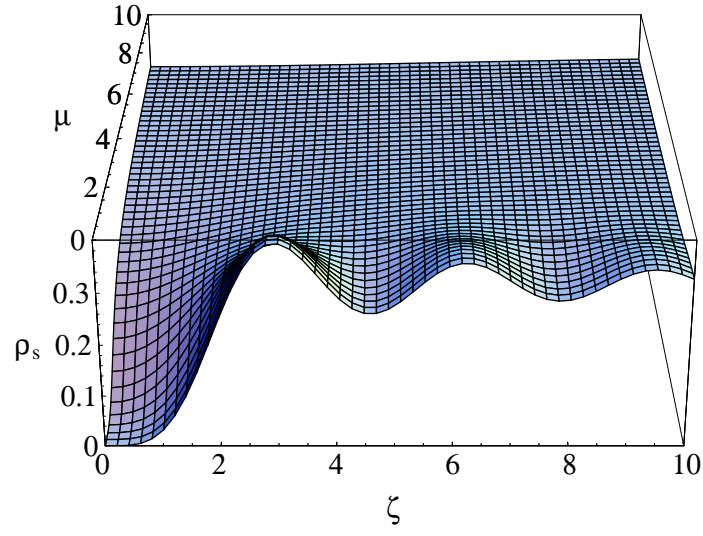
FIG. 3. The scaled spectral density $\rho_s(\zeta; \mu, \mu, -\mu, -\mu)$ for the symplectic ensemble with four flavors ($\alpha = 1$).
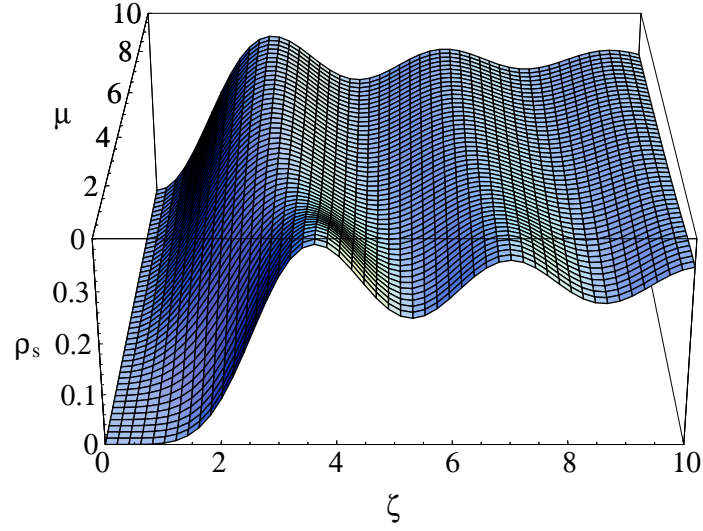

FIG. 4. The scaled spectral density $\rho_s(\zeta; \mu, \mu, -\mu, -\mu, 0, 0)$ for the symplectic ensemble with six flavors ($\alpha = 1$).